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Free field realization of superstring theory on AdS_3

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Abstract: The Coulomb gas representation of expectation values in SU(2) conformal field theory developed by Dotsenko is extended to the SL$(2,\mathbb{R})$ WZW model and applied to bosonic string theory on AdS_3 and to Type II superstrings on $AdS_3 \times \mathcal{N}$. The spectral flow symmetry is included in the free field realization of vertex operators creating superstring states of both Ramond and Neveu-Schwarz sectors. Conjugate representations for these operators are constructed and a background charge prescription is designed to compute correlation functions. Two and three point functions of bosonic and fermionic string states in arbitrary winding sectors are calculated. Scattering amplitudes that violate winding number conservation are also discussed.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

Two and three dimensional toy models of string theory have been useful to explore some essential features of theoretical physics in a setting with a vastly reduced number of dynamical degrees of freedom. Particularly interesting examples can be found in nonperturbative physics, the continuation to lorentzian signature, the notion of time in curved backgrounds, singularities and conceptual problems of black hole physics.

One of the simplest frames where these questions can be addressed is the $\mathrm{SL}(2,\mathbb{R})$ group manifold. The WZW model on $\mathrm{SL}(2,\mathbb{R})$ is an exact conformal field theory describing string propagation in three dimensional Anti de Sitter spacetime (AdS_3) [1]. This model is closely related to two and three dimensional black holes in string theory through the

SL$(2,\mathbb{R})/$ U(1) coset [2] and orbifolding [3] respectively. SL$(2,\mathbb{R})$ cosets are also linked to Liouville theory and some of its generalizations [4, 5] and to the physics of defects or singularities [6, 7].

Despite its simplicity, the efforts to develop a consistent string theory on AdS_3 turned out to be highly non trivial. The origin of the difficulties can be traced to the non-compact nature of SL$(2,\mathbb{R})$ and the non-rational structure of the worldsheet CFT [8]. The resolution of the main problem, the apparent lack of unitarity of the theory, was possible with the help of the AdS/CFT duality conjecture [9]. The impact of the conjecture was twofold: on the one hand this is an example where the AdS/CFT duality has been explored beyond the supergravity approximation, with complete control over the theory in the bulk [10, 11]; on the other hand, the conjecture provided a productive feedback on the interpretation of the puzzles raised by the worldsheet theory [12].

The structure of the Hilbert space of the SL$(2,\mathbb{R})$ WZW model was determined in [13] where the spectrum of physical states of string theory on AdS_3 and a proof of the no-ghost theorem were given as well. It was realized that the model has a spectral flow symmetry which gives rise to new representations for the string spectrum besides the standard discrete and continuous unitary series which had been considered previously [8]. The computation of the one loop partition function performed in [14, 15], provided further evidence for the spectrum of the free theory.

To establish the consistency of the full theory one has to consider interactions and verify the closure of the operator product expansion. But the fusion rules are difficult to find in the non-compact worldsheet CFT that defines string theory on AdS_3 because there are generically no null vectors in the relevant current algebra representations, so that most of the techniques from rational conformal field theories are not available and consequently the factorization properties of the model have not been completely determined yet. Nevertheless, important progress has been achieved recently in the resolution of this problem. Correlation functions of primary fields have been calculated using different procedures in the euclidean version of the theory, the $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)} \equiv H_3^+$ model. The path integral method to obtain expectation values was started in [16] and applied to the computation of two and three point amplitudes of bosonic string states in [17]. A generalization of the bootstrap approach was designed by Teschner and some four point functions were given in [18]. The physical interpretation of these exact results was performed by Maldacena and Ooguri in [12] where correlation functions involving spectral flowed operators were also presented.

Scattering amplitudes of $n-$ states in string theory on AdS_3 exhibit several subtleties for $n \geq 3$. On the one hand, correlation functions of discrete states are only well defined if the sum of the isospins j of the external operators satisfies $\sum_i j_i < k-3$ (where k is the level of the Kac Moody algebra). Moreover the four point functions do not factorize as expected into a sum of products of three point functions with physical intermediate states unless the quantum numbers of the external states verify $j_1 + j_2 < \frac{k-3}{2}$ and $j_3 + j_4 < \frac{k-3}{2}$. The meaning of these constraints was proposed in [12]: correlation functions violating these bounds do not represent a well-defined computation in the dual CFT$_2$. However one would like to better understand this unusual feature from the worldsheet viewpoint.

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On the other hand, a curious aspect of this model is that physical amplitudes of n string states may violate winding number conservation up to $n-2$ units. This fact is well understood from the representation theory of $\mathrm{SL}(2,\mathbb{R})$ [12]. Nevertheless the computation of winding non-conserving scattering amplitudes proposed in [12] involves the insertion of spectral flow operators in the correlation functions. This implies the computation of expectation values of more vertex operators than the n original ones. This procedure has been applied to three point functions, but four point functions violating winding number conservation by one or two units require the calculation of correlators with five or six operator insertions, with the consequent complications. These amplitudes are needed to definitely establish the unitarity of the theory through the analysis of their factorization properties.

Therefore it seems necessary to develop techniques that simplify these computations and allow to perform others that would clarify the full structure of the model. The free field description of the theory appears as a powerful tool in this direction. This approximation was initially applied in [10, 11] to derive the spacetime CFT and establish the conjectured AdS/CFT correspondence in the three dimensional case (for related work see [19]). Even though this approach is expected to give a good picture of the theory only near the boundary of AdS_3, the computation of two and three point amplitudes of string states using the Coulomb gas formalism in [20, 21] has produced results in complete agreement with the exact ones. Moreover, the analysis of unitarity in this approximation might give important information on the consistency of the complete theory. For that reason, the aim of this article is to further develop this approach and extend the formulation to the supersymmetric case.

The Coulomb gas formalism to compute conformal blocks in the compact SU(2) CFT was introduced by Dotsenko and applied to the computation of conformal blocks for integer $2j$ and admissible representations in [22]. The non-compact $\mathrm{SL}(2,\mathbb{R})$ case was considered in [23, 24] where the degenerate case was resolved. Two and three point amplitudes of physical string states in the coset theory $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ were computed in [25]. More recently it was extended to take into account the spectral flow symmetry of $\mathrm{SL}(2,\mathbb{R})$ in [20] and applied to the computation of two and three point functions of AdS_3 string states for arbitrary winding sectors, both preserving and violating winding number conservation, in [21].

Here we study superstring theory on $AdS_3 \times \mathcal{N}$. We consider vertex operators both in the Ramond and Neveu-Schwarz sectors in the free field approximation and construct their conjugate representations. We develop Dotsenko's background charge prescription to compute expectation values in the supersymmetric theory and employ it to calculate scattering amplitudes of two and three superstring states. We analyze the structure of winding (non)conservation pattern and discuss the relevance of the internal theory concerning this question.

The organization of the paper is as follows. In section 2 the Coulomb gas representation of correlators in bosonic string theory on AdS_3 is reviewed and completed. In section 3 the realization of the supersymmetric theory is presented and correlation functions of two and three superstring states are computed, both in the Ramond and Neveu Schwarz sectors of the theory. Conclusions are presented in section 4.

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2. Bosonic string theory on AdS_3

In this section we briefly review string theory on AdS_3 mainly to introduce our notation and conventions. We complete the Coulomb gas formulation of expectation values of string states started in [20, 21].

The metric of euclidean AdS_3 (the hyperbolic space H_3^+) can be written in *Poincaré* coordinates as

$$ds^2 = l^2(d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma}) , \tag{2.1}$$

where $\phi \in \mathbb{R}$, $\{\gamma, \bar{\gamma}\}$ are complex coordinates parametrizing the boundary of H_3^+ which is located at $\phi \to \infty$ and the parameter l is related to the scalar curvature as $\mathcal{R} = -2/l^2$.

Consistent string propagation in this background metric requires in addition an antisymmetric rank two tensor background field $B = l^2 e^{2\phi} d\gamma \wedge d\bar{\gamma}$. The theory is described by a non linear sigma model with action

$$S = \frac{k}{8\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}) , \tag{2.2}$$

where $k = l^2/l_s^2$ and l_s^2 is the fundamental string length, which is equivalent to a WZW model on $\mathrm{SL}(2,\mathbb{R})$ (or actually its euclidean version $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$). This action has a larger symmetry than the isometries of the group, namely $g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z})$, with $g, \Omega \in \mathrm{SL}(2,\mathbb{R})$. The corresponding currents $J(z) = -\frac{k}{2}(\partial g)g^{-1}, \bar{J}(\bar{z}) = -\frac{k}{2}(\bar{\partial} g^{-1})g$ can be expanded in Laurent series

$$J^a(z) = \sum_{n=-\infty}^{\infty} J_n^a \; z^{-n-1} \tag{2.3}$$

and the coefficients J_n^a satisfy a Kac-Moody algebra given by

$$[J_n^a, J_m^b] = i\epsilon_c^{ab} J_{n+m}^c - \frac{k}{2}\eta^{ab} n \delta_{n+m,0} , \tag{2.4}$$

where the Cartan Killing metric is $\eta^{+-} = \eta^{-+} = 2$, $\eta^{33} = -1$ and ϵ_c^{ab} is the Levi Civita antisymmetric tensor. And similarly for the antiholomorphic currents.

The Sugawara stress-energy tensor is given by

$$T = \frac{\eta_{ab}}{k-2} : J^a(z) J^b(z) : \tag{2.5}$$

It is related to the Casimir of the group as $C = (k-2)T$ and it leads to the following central charge of the Virasoro algebra

$$c = \frac{3k}{k-2} = 3 + \frac{12}{\alpha_+^2} , \tag{2.6}$$

($\alpha_+ = \sqrt{2(k-2)}$).

The classical solutions of this theory were presented in [13]. Timelike geodesics oscilate around the center of AdS_3 whereas spacelike geodesics representing tachyons travel from one side of the boundary to the opposite. Solutions describing string propagation are obtained from the dynamics of pointlike particles through the spectral flow operation. Timelike geodesics give rise to *short strings*, bound states trapped in the gravitational potential of AdS_3. Conversely, *long strings* arising from spacelike geodesics can reach the boundary of AdS_3. The spectral flow parameter w is an integer named *winding number*. Different values of w correspond to distinct solutions, even at the classical level (as exhibited, for instance, by the energy spectrum).

At the quantum level, the building blocks of the Hilbert space $\mathcal{H}$ are unitary hermitic representations of $\mathrm{SL}(2,\mathbb{R})$. The states $|j,m>$ satisfy

$$\begin{aligned} C_0|j,m\rangle &= j(j+1)|j,m\rangle\,, \qquad J_0^3|j,m\rangle = m|j,m\rangle, \\ J_0^{\pm}|j,m\rangle &= (m\mp j)|j,m\pm 1\rangle\,, \end{aligned} \tag{2.7}$$

with

$$\{m\in\mathbb{R}, j\in\mathbb{R}\} \quad \vee \quad \left\{m\in\mathbb{R}, j\in -\frac{1}{2}+i\mathbb{R}\right\} \tag{2.8}$$

as required by hermiticity and in addition they must be Kac Moody primaries, namely

$$J_n^a|j,m\rangle = 0 \quad \forall\, n>0\,. \tag{2.9}$$

The allowed representations are:

- Discrete lowest weight representation

$$\mathcal{D}_j^+ = \{|j,m\rangle;\;\; j\in\mathbb{R};\;\;\; m=j+1, j+2, j+3,\ldots\} \tag{2.10}$$

- Discrete highest weight representation

$$\mathcal{D}_j^- = \{|j,m\rangle;\;\; j\in\mathbb{R};\;\;\; m=-j-1,-j-2,-j-3,...\} \tag{2.11}$$

- Principal continuous representation

$$\mathcal{C}_j^{\alpha} = \{|j,m\rangle;\;\; j=-\frac{1}{2}+i\lambda;\; \lambda\in\mathbb{R};\;\;\; m=\alpha,\alpha\pm 1,\alpha\pm 2,\ldots;\;\;\; \alpha\in\mathbb{R}\} \tag{2.12}$$

For applications to string theory one considers the universal cover of $\mathrm{SL}(2,\mathbb{R})$, where j is not quantized. Notice that the vectors in $\mathcal{H}$ related by $j\leftrightarrow -1-j$ represent the same physical state and therefore j can be restricted to $j\geq -1/2$. The complete basis of $\mathcal{L}^2(AdS_3)$ is given by $\mathcal{C}^{\alpha}_{j=-1/2+i\lambda}\times\mathcal{C}^{\alpha}_{j=-1/2+i\lambda}$ and $\mathcal{D}_j^{\pm}\times\mathcal{D}_j^{\pm}$ with $j>-1/2$.

The representation space can be enlarged by acting on the primary states in these series with J_n^a, $n<0$. The corresponding representations are denoted by $\widehat{\mathcal{D}}_j^{\pm}, \widehat{\mathcal{C}}_j^{\alpha}$. Furthermore the full representation space contains the spectral flow images of these series which correspond to winding classical strings. Actually the spectral flow operation leads to the following automorphism of the $\mathrm{SL}(2,\mathbb{R})$ currents

$$J_n^3 \to \widetilde{J}_n^3 = J_n^3 - \frac{k}{2}w\delta_{n,0} \tag{2.13}$$

$$J_n^{\pm} \to \widetilde{J}_n^{\pm} = J_{n\pm w}^{\pm} \tag{2.14}$$

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with $w \in \mathbb{Z}$ and consequently the modes of the Virasoro generators transform as

$$L_n \to \widetilde{L}_n = L_n + wJ_n^3 - \frac{k}{4}w^2\delta_{n,0}\,. \tag{2.15}$$

Unlike the compact SU(2) case, the new operators generate inequivalent representations of $\mathrm{SL}(2,\mathbb{R})$ with states $|\tilde{j},\tilde{m},\omega\rangle$ satisfying

$$\widetilde{L}_0|\tilde{j},\widetilde{m},w\rangle = -\frac{\tilde{j}(\tilde{j}+1)}{k-2}|\tilde{j},\widetilde{m},w\rangle\,, \quad \widetilde{J}_0^3|\tilde{j},\widetilde{m},w\rangle = \widetilde{m}|\tilde{j},\widetilde{m},w\rangle \tag{2.16}$$

Finally, the complete Hilbert space of string theory on AdS_3 is obtained by applying creation operators $\widetilde{J}_n^a, n < 0$ on the primary states defined by (2.16) and verifying the physical state conditions

$$(L_0-1)|\tilde{j},\widetilde{m},w,\widetilde{N},h\rangle = \left(-\frac{\tilde{j}(\tilde{j}+1)}{k-2} - w\widetilde{m} - \frac{k}{4}w^2 + \widetilde{N} + h - 1\right)|\tilde{j},\widetilde{m},w,\widetilde{N},h\rangle = 0 \tag{2.17}$$

$$L_n|\tilde{j},\widetilde{m},w,\widetilde{N},h\rangle = \left(\widetilde{L}_n - w\widetilde{J}_n^3\right)|\tilde{j},\widetilde{m},w,\widetilde{N},h\rangle = 0 \quad \text{for } n>0\,, \tag{2.18}$$

where $\widetilde{N}$ is the excitation level of $\widetilde{J}_n$ and h is the conformal weight of the state in the internal theory.[1]

Notice that the representations $\widehat{\mathcal{D}}_{\tilde{j}}^{\pm,w=\mp1}$ and $\widehat{\mathcal{D}}_{\frac{k}{2}-2-\tilde{j}}^{\mp,w=0}$ are equivalent. This has an important consequence on the values allowed for j. Indeed, recalling the symmetry $j \leftrightarrow -1-j$ which implies $j \geq -1/2$, j is restricted as required by the no-ghost theorem [13] to

$$-\frac{1}{2} < j < \frac{k-3}{2}\,. \tag{2.19}$$

2.1 Free field representation of string theory on AdS_3

The free field formulation of this theory follows from the action (2.2) which can be rewritten as a free field model by introducing auxiliary fields $\beta, \bar{\beta}$ as

$$S = \frac{k}{8\pi}\int \mathrm{d}^2z \left(\partial\phi\overline{\partial}\phi + \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} - \beta\overline{\beta}e^{-2\phi}\right). \tag{2.20}$$

Quantization leads to include some renormalization factors [27] as

$$S = \frac{1}{4\pi}\int \mathrm{d}^2z \left(\partial\phi\overline{\partial}\phi - \frac{2}{\alpha_+}R\phi + \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} - \beta\overline{\beta}e^{-\frac{2}{\alpha_+}\phi}\right), \tag{2.21}$$

where R is the scalar curvature of the worldsheet. The interaction term $\beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi}$ becomes negligible near the boundary ($\phi \to \infty$) and the theory can thus can be treated perturbatively in this region. It can be fully described in terms of OPEs of free fields, namely

$$\phi(z)\phi(z') \sim -\ln(z-z')\,, \qquad \gamma(z)\beta(z') \sim -\frac{1}{z-z'}\,. \tag{2.22}$$

[1] We have been considering string theory on AdS_3, but more generally we could take a background $AdS_3 \times \mathcal{N}$, with $\mathcal{N}$ a compact internal manifold. An interesting example has been considered recently in relation to string amplitudes in the plane wave limit of $AdS_3 \times S^3$ in [26].

The currents are defined in the *Wakimoto* representation [28] as

$$J^+(z) \equiv -\beta(z) \tag{2.23}$$

$$J^3(z) \equiv -\beta(z)\gamma(z) - \frac{\alpha_+}{2}\partial\phi(z) \tag{2.24}$$

$$J^-(z) \equiv -\beta(z)\gamma^2(z) - \alpha_+\gamma(z)\partial\phi(z) - k\partial\gamma(z) \tag{2.25}$$

and the energy momentum tensor is

$$T(z) = \beta(z)\partial\gamma(z) - \frac{1}{2}\partial\phi(z)\partial\phi(z) - \frac{1}{\alpha_+}\partial^2\phi(z)\,. \tag{2.26}$$

It is easy to see that the $\beta\gamma$ fields form a commuting $bc-$system with conformal weight 1 (β) and 0 (γ) and ghost charge 1.

The currents satisfy the OPEs

$$J^+(z)J^-(z') \sim \frac{k}{(z-z')^2} - \frac{2J^3(z')}{z-z'} \tag{2.27}$$

$$J^3(z)J^\pm(z') \sim \pm\frac{J^\pm(z')}{z-z'} \tag{2.28}$$

$$J^3(z)J^3(z') \sim -\frac{\frac{k}{2}}{(z-z')^2} \tag{2.29}$$

in full agreement with the commutation relations (2.4).

2.2 Vertex operators

It is now possible to define the vertex operators representing string states. We shall deal with operators in the free field approximation. For a detailed analysis of the exact theory see [12, 13].

In general one works on $AdS_3 \times \mathcal{N}$ where the vertex operators factorize as $V_{AdS_3\times\mathcal{N}} = V_{AdS_3} \times V_{\mathcal{N}}$. In the remaining of this section we shall consider only the AdS_3 part of the vertex operators. In the zero winding sector they may be written as

$$V_{AdS_3} = V_{j,m,\overline{m}} = \gamma^{j-m}\overline{\gamma}^{j-\overline{m}}e^{\frac{2j}{\alpha_+}\phi}\,, \tag{2.30}$$

where j,m must belong to either $\mathcal{D}_j^\pm$ or $\mathcal{C}^\alpha_{-\frac{1}{2}+i\lambda}$ and $m-\overline{m}\in\mathbb{Z}$ is required by singlevaluedness on the spacetime coordinates $\{\gamma,\overline{\gamma}\}$. This condition will arise more formally in the next section after introducing the spectral flow operators [13, 49].

The vertex operator (2.30) has the following OPEs with the currents (2.23)–(2.25)

$$J^+(z)V_{j,m}(z') \sim (m-j)\frac{V_{j,m+1}(z')}{z-z'} \tag{2.31}$$

$$J^3(z)V_{j,m}(z') \sim m\frac{V_{j,m}(z')}{z-z'} \tag{2.32}$$

$$J^-(z)V_{j,m}(z') \sim (m+j)\frac{V_{j,m-1}(z')}{z-z'} \tag{2.33}$$

as required for a Kac Moody primary state. Excited string states can be constructed from these ones by acting with creation modes of the currents. The conformal weight of the operators (2.30) can be read from

$$T(z)V_{j,m}(z') \sim \frac{-j(j+1)}{k-2}\frac{V_{j,m}(z')}{(z-z')^2} + \frac{\partial V_{j,m}(z')}{z-z'} \,. \tag{2.34}$$

Therefore any primary state in the zero winding sector can be represented by (2.30). How can one represent states in arbitrary winding sectors?

Two proposals can be found in the literature. One of them relies on the bosonization of the $\beta\gamma$-system followed by a redefinition of the scalars [29, 30]. The winding number appears naturally in this realization after compactifying one of the light-like coordinates [31] (recall that $\gamma, \bar{\gamma}$ parametrize the boundary of AdS_3 which is compact in the angular direction). The other approach implies the factorization $\mathrm{SL}(2,\mathbb{R}) \to \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$, as suggested by the no-ghost theorem [32]. This proposal, that we shall follow, arises naturally in the supersymmetric theory (see section 3).

The strategy to introduce winding in the product theory $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$ is to first gauge the timelike U(1) current corresponding to the J^3 generator of $\mathrm{SL}(2,\mathbb{R})$. This gives an euclidean theory representing a two dimensional black hole [2, 33]. Since one is gauging a compact U(1) the winding number arises as a restriction on the allowed values of $m+\overline{m}$. However this condition disappears when adding back a non-compact J^3 current. Indeed this current can be appended in any winding sector, thus introducing a missmatch with the gauged U(1) current. This procedure allows to realize the currents and vertex operators in arbitrary winding sectors [34].

To gauge the U(1) current from $\mathrm{SL}(2,\mathbb{R})$ one introduces the fields $A(z)$ and $\overline{A}(\overline{z})$ which, after choosing a gauge slice, can be represented in terms of a free scalar field X as $A = -\partial X$, $\overline{A} = -\overline{\partial} X$ [33] and $X(z)X(w) \sim -\ln(z-w)$. Choosing a particular gauge produces a jacobian that can be realized by a fermionic $bc-$system with fields $B(z)$ and $C(z)$ having weights 1 and 0 respectively. As usual when fixing the gauge there is a $BRST$ charge which must commute with the states of the theory. In this case one obtains

$$Q^{\mathrm{U}(1)} = \oint C(z)\left(J^3(z) - i\sqrt{\frac{k}{2}}\partial X(z)\right) \mathrm{d}z \,. \tag{2.35}$$

The holomorphic part of the vertex operators in this coset theory can be naturally written as

$$V_{j,m}^{\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}} = V_{j,m} e^{i\sqrt{\frac{2}{k}}mX} \,. \tag{2.36}$$

Now we have to reintroduce the J^3 current. The OPE (2.29) suggests the following bosonization

$$J^3(z) \equiv -i\sqrt{\frac{k}{2}}\partial Y(z) \,, \tag{2.37}$$

where $Y(z)$ is a scalar field with timelike signature (recall we are working in euclidean AdS_3 and thus $Y(z)Y(w) \sim +\ln(z-w)$).

Finally the full energy momentum tensor is

$$T \equiv T_{\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times \mathrm{U}(1)} = \beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{1}{\alpha_+}\partial^2\phi - \frac{1}{2}\partial X\partial X - B\partial C + \frac{1}{2}\partial Y\partial Y \tag{2.38}$$

and the vertex operators can be written as

$$V_{j,m,p} = V_{j,m} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}pY} . \tag{2.39}$$

Note that in this case there is no a priori connection between the quantum numbers p (corresponding to the U(1) theory) and m (corresponding to $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$). This is what allows to include the winding number. Even though p does not depend on m directly, the states represented by (2.39) must correspond to unitary representations of $\widehat{SL}(2,\mathbb{R})_k$. Therefore p is restricted to $p = m + \frac{k}{2}w$, according to (2.13).[2] Therefore the final form of the vertex operators is

$$V_{j,m,w} = V_{j,m} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}w)Y} = \gamma^{j-m} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}w)Y} . \tag{2.40}$$

It is easy to check that the conformal dimension of $V_{j,m,w}$ is as expected from (2.17), namely

$$\Delta(V_{j,m,w}) = -\frac{j(j+1)}{k-2} - mw - \frac{kw^2}{4} . \tag{2.41}$$

Observe that the quantum numbers obtained by applying the currents (2.23)–(2.25) to the vertex operators (2.40) coincide with those produced by applying them to (2.30). This indicates that (2.23)–(2.25) corrrespond to the tilded currents, acting like (2.16), and thus the quantum numbers in $V_{j,m,w}$ are actually tilded variables. What is the correct realization of the original currents?

It is easy to verify that the following definitions satisfy the algebra and produce the correct quantum numbers when acting on the vertex operators (2.40)

$$J^+(z) \equiv -\beta(z) e^{i\sqrt{\frac{2}{k}}(X(z)+Y(z))} \tag{2.42}$$

$$J^-(z) \equiv -\left(\beta(z)\gamma^2(z) + \alpha_+\gamma(z)\partial\phi(z) + k\partial\gamma(z)\right) e^{-i\sqrt{\frac{2}{k}}(X(z)+Y(z))} \tag{2.43}$$

$$J^3(z) \equiv -i\sqrt{\frac{k}{2}}\partial Y . \tag{2.44}$$

2.2.1 Spectral flow operators

As mentioned above there is a formalism where the restriction $m - \overline{m} \in \mathbb{Z}$ appears naturally [49]. Moreover the construction provides a method to obtain the vertex operators in $w \neq 0$ sectors from those in $w = 0$. One advantage of this mechanism is that it allows to introduce winding number very easily in the supersymmetric vertex operators and thus we review it here.

The spectral flow operator in the theory on the product $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$ is defined as

$$\mathcal{F}^w(z,\overline{z}) = \mathcal{F}^w(z)\cdot\overline{\mathcal{F}}^w(\overline{z}) = e^{iw\sqrt{\frac{k}{2}}\left(Y(z)+\overline{Y}(\overline{z})\right)} . \tag{2.45}$$

[2] Notice that m here is $\widetilde{m}$ in (2.16). We drop the tildes to lighten the notation.

Locality and closure of the OPEs are two important consistency requirements. In particular the following OPE

$$\begin{aligned}\mathcal{F}^w(z,\overline{z})V_{j,m,\overline{m},w=0}(z',\overline{z}') &\sim (z-z')^{-wm}(\overline{z}-\overline{z}')^{-w\overline{m}}V_{j,m,\overline{m},w}(z',\overline{z}')\\ &= (z-z')^{-(m-\overline{m})w}|z-z'|^{-2\overline{m}w}V_{j,m,\overline{m},w}(z',\overline{z}')\end{aligned} \tag{2.46}$$

implies that the operators $V_{j,m,\overline{m},w}$ must be included in the theory and $m-\overline{m}\in\mathbb{Z}$. It may be verified that $V_{j,m,\overline{m},w}$ coincide with (2.40).

2.3 Correlation functions and the Coulomb gas formalism

The Coulomb gas formalism to compute correlation functions was found to be very natural to obtain scattering amplitudes violating winding number conservation in [20, 21]. Here we briefly review the basic features of those works and in the next section we develop the supersymmetric extension.

Correlation functions are defined as usual through an euclidean functional integral, namely

$$\Big\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\Big\rangle_\Sigma \equiv \int [\mathrm{d}\phi]\, e^{-S}\, V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\,, \tag{2.47}$$

where $V_{\alpha_i}(z_i)$ are the vertex operators (2.40) with quantum numbers $\alpha_i = j_i, m_i, \bar{m}_i, w_i$, Σ denotes the compact topology of the worldsheet (here we shall work on the sphere) and the action S is given by (2.21). The measure $[\mathrm{d}\phi]$ is a compact notation for the measure of all fields involved. This formalism allows to compute correlators as a perturbative expansion in the interaction term

$$S_{\text{int}} = \frac{1}{4\pi}\int \mathrm{d}^2z\beta\overline{\beta}e^{-\frac{2}{\alpha_+}\phi}\,. \tag{2.48}$$

Scattering amplitudes are obtained from (2.47) after integrating the insertion points of the vertex operators over the complex plane and dividing by the volume of the conformal group as

$$\mathcal{A}_{\alpha_1\ldots\alpha_n} = \frac{1}{\mathrm{Vol}_{\mathrm{PSL}(2,\mathbb{C})}}\int \mathrm{d}^2z_1\ldots\mathrm{d}^2z_n\Big\langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\Big\rangle_{S^2}\,. \tag{2.49}$$

Since the action is free except for the factor S_{int}, it is also possible to define the correlators through Wick contractions. The perturbative expansion of the functional integral is thus reproduced by inserting powers of S_{int} into the correlators. We follow this purely algebraic procedure (which does not rely on the action once the propagators are given) because it provides a natural way to introduce the Coulomb gas formalism.

There are basically two types of correlators to compute: those involving exponentials of free fields (ϕ, X, Y) and those containing $\beta\gamma$ fields. As mentioned above these last ones form a $bc-$system with background charge 1. They can be bosonized as $\beta\cong -i\partial v e^{iv-u}$, $\gamma\cong e^{u-iv}$ where u and v are canonically normalized bosons with background charge 1 and $-i$ respectively. Therefore one only has to consider exponential operators of free fields, eventually with a background charge Q.[3]

[3]The term ∂v in the bosonization of β can be written as $\beta\cong -\left(\partial e^{iv}\right)e^{-u}$, thus one can perform Wick contractions of exponential factors in this case as well and apply the operator ∂ at the end of the calculation.

As usual nonvanishing correlators must satisfy the conservation law $Q + \sum_{i=1}^{n} \alpha_i = 0$. This raises a problem for the two point functions of a vertex operator with itself which is in general expected to be nonvanishing. The solution is provided by the conjugate vertex operators $\widetilde{V_\alpha} = V_{-Q-\alpha}$. The general solution for higher point functions was developed by Dotsenko and Fateev in [35] and the strategy is to introduce the so called *screening operators* in the correlation functions. These insertions must not alter the conformal structure of the correlators and therefore they must commute with the currents and have zero conformal dimension. These observations lead to consider the following non local operators [36, 37]:

$$\mathcal{S}_+ = \int \mathrm{d}^2 z\, \beta(z)\overline{\beta}(\overline{z}) e^{-\frac{2}{\alpha_+}\phi(z,\overline{z})} \quad ; \quad \mathcal{S}_- = \int \mathrm{d}^2 z\, \beta(z)^{\frac{\alpha_+^2}{2}} \overline{\beta}(\overline{z})^{\frac{\alpha_+^2}{2}} e^{-\alpha_+\phi(z,\overline{z})} . \tag{2.50}$$

Consequently the correlators in string theory on AdS_3 can be written as[4]

$$\left\langle \mathcal{S}_+^{n_+} \mathcal{S}_-^{n_-} V_{j_1,m_1,w_1}(z_1) \cdots V_{j_n,m_n,w_n}(z_n) \right\rangle_{S^2} \tag{2.51}$$

and the conservation laws are

$$\beta\gamma : \#\gamma - \#\beta + Q_{\beta\gamma} = 0 \quad \rightarrow \quad \sum_{i=1}^{n} (j_i - m_i) - n_+ - \frac{\alpha_+^2}{2} n_- + 1 = 0 \tag{2.52}$$

$$\phi : \sum_i \alpha_i^\phi + Q_\phi = 0 \quad \rightarrow \quad \frac{2}{\alpha_+}\left(\sum_{i=1}^{n} j_i - n_+ - \frac{\alpha_+^2}{2} n_- + 1 \right) = 0 \tag{2.53}$$

$$X : \sum_i \alpha_i^X = 0 \quad \rightarrow \quad i\sqrt{\frac{2}{k}} \sum_{i=1}^{n} m_i = 0 \tag{2.54}$$

$$Y : \sum_i \alpha_i^Y = 0 \quad \rightarrow \quad i\sqrt{\frac{2}{k}} \sum_{i=1}^{n} \left(m_i + \frac{k}{2} w_i \right) = 0 , \tag{2.55}$$

where α_i represent the charge of the operators under the various fields. Equation (2.52) is contained in the other three, thus they can be summarized as

$$\sum_{i=1}^{n} j_i + 1 = n_+ + (k-2) n_- ; \qquad \sum_{i=1}^{n} m_i = 0 ; \qquad \sum_{i=1}^{n} w_i = 0 , \tag{2.56}$$

where the quantum numbers can be read from the vertex operators $V_{j,m,w}$. In the non-compact theory $-\frac{1}{2} < j < \frac{k-3}{2}$ for the discrete series and $j = -\frac{1}{2} + i\lambda$ for the principal continuous series. Therefore it is necessary to consider the analytic extension of (2.56) for $n_+, n_- \in \mathbb{C}$. Actually, once this generalization is allowed any correlator can be computed using only one kind of screening operators.

Similarly as in the case of minimal models it is possible to define conjugate operators in the $\mathrm{SL}(2,\mathbb{R})$ WZW model. One candidate for conjugate operator to $V_{j,m,w}$ is $\widetilde{V_{j,m,w}} = V_{-1-j,m,w}$.[5] Indeed this operator has the correct conformal dimension and OPE with the

[4]Notice that $\mathcal{S}_+$ is the interaction term in the action (2.48), therefore computing the correlators (2.51) using $n_- = 0$ is completely equivalent to a perturbative expansion of order n_+ in the path integral formalism.

[5]Strictly, one has to multiply $V_{-1-j,m,w}$ by a coefficient proportional to $\frac{\Gamma(j+1-m)}{\Gamma(-j-m)}$. In fact, this coefficient is nothing else but the two point function. This is related to the Fourier expansion of the square integrable functions on AdS_3 [10].

currents. One can verify also that the two point functions $\langle V_{j_1,m_1,w_1} V_{-1-j_2,m_2,w_2}\rangle$ do not require screening operators if $j_2 = j_1, m_2 = -m_1$ and $w_2 = -w_1$ (see (2.56)). The signs in m and w refer to the distinction between *ingoing* and *outgoing* states.[6]

The formalism reviewed above allows in principle to compute any correlation function satisfying winding number conservation. However it was suggested in [38] and shown in [12] that n-point functions violating winding number conservation up to $n-2$ units can be in general nonvanishing. This was considered in the free field approximation in [20, 21] where the algebraic formulation was used to introduce new conservation laws, thus extending the original idea designed by Dotsenko [22].

To implement this procedure it is important to consider different representations of the identity operator. The identity has zero conformal weight and regular OPE with the currents. These conditions are satisfied by the state $|j,m,w\rangle = |-1,0,0\rangle$ (notice that $m=0$ singles out $j=-1$ over $j=0$ if this state is to belong to one of the discrete series (2.11)–(2.12)). This implies that the identity is not a physical state.

The first non trivial representation of the identity one can consider is the conjugate operator[7] $\widetilde{1} \equiv \mathcal{I}_{-1} = V_{-1,0,0} = \gamma^{-1} e^{\frac{-2}{\alpha_+}\phi}$. However this realization is obtained when conjugating with respect to the conservation laws (2.56) and then one cannot expect that it solves the winding nonconservation problem.

There is another well known representation of the identity given by the operator $\widetilde{\mathcal{I}}_0 = \beta^{k-1} e^{\frac{2(1-k)}{\alpha_+}\phi}$ [22]. It leads to new conservation laws assuring that $\left\langle \widetilde{\mathcal{I}}_0\, 1\right\rangle_{S^2}$ is non vanishing. Notice that redefining the conjugate identity (from $\mathcal{I}_{-1}$ to $\widetilde{\mathcal{I}}_0$) is equivalent to redefining the out vacuum of the theory. The corresponding conservation laws are

$$\beta\gamma[\widetilde{\mathcal{I}}_0] : \#\gamma - \#\beta = 1-k \tag{2.57}$$

$$\phi[\widetilde{\mathcal{I}}_0] : \sum_i \alpha_i^\phi = \frac{2(1-k)}{\alpha_+} \tag{2.58}$$

$$X[\widetilde{\mathcal{I}}_0] : \sum_i \alpha_i^X = 0 \tag{2.59}$$

$$Y[\widetilde{\mathcal{I}}_0] : \sum_i \alpha_i^Y = 0\,. \tag{2.60}$$

It is interesting to note that introducing one screening operator $\mathcal{S}_-$ in the correlation functions deduced from $\widetilde{\mathcal{I}}_0$ one obtains the original conservation laws (2.52)–(2.55). This suggests that it is possible to go from one case to the other redefining the out vacuum through the inclusion of a screening operator $\mathcal{S}_-$. This observation indicates that all the correlation functions computed using (2.57)–(2.60), can also be computed using (2.52)–

[6] One could have included these signs in the definition of conjugation; we choose not to do that in order to stress the conceptual idea that both operators represent the same physical state.

[7] The free field formalism is, in principle, valid near the boundary. This means that, because $j=-1$, one would have to use the conjugate vertex $\widehat{V_{-1,0,0}}$, that dominates in the limit $\phi \to \infty$. Indeed, this prescription gives the usual identity operator 1. The existence of screening operators in the $w=0$ sectors indicates that it is also possible to obtain a conjugate identity under $j \leftrightarrow -j-1$.

(2.55). Therefore this new representation cannot solve the problem of winding non conservation either. There is a completely analogous statement that uses $\mathcal{S}_+$ to relate the usual identity with $\mathcal{I}_{-1}$.

The identities which solve the problem can be obtained recalling the equivalence $\widehat{\mathcal{D}}_{\tilde{j}}^{\pm,w=\mp 1} \sim \widehat{\mathcal{D}}_{\frac{k}{2}-2-\tilde{j}}^{\mp,w=0}$. Indeed, an identity allowing to violate winding conservation must belong to one of the $w \neq 0$ sectors, and the equivalence between representations in $w=0$ and $w=\pm 1$ assures the existence of such operator. Actually the replacement $j \to \frac{k}{2}-2-j$ with $j=-1$ in[8] $\widetilde{V_{j,m,w}}$ leads to the following operators[9]

$$\mathcal{I}_+ = e^{-\frac{k}{\alpha_+}\phi} e^{-i\sqrt{\frac{k}{2}}X} \tag{2.61}$$

$$\mathcal{I}_- = \gamma^{-k} e^{-\frac{k}{\alpha_+}\phi} e^{i\sqrt{\frac{k}{2}}X}\,. \tag{2.62}$$

They both have zero conformal weight and commute with J^3. $\mathcal{I}_+$ ($\mathcal{I}_-$) commutes with J^- (J^+) whereas the residue of the OPE with J^+ (J^-) is a spurious state which decouples in the correlators.

The conservation laws associated to $\mathcal{I}_+$ are

$$\beta\gamma[\mathcal{I}_+] : \#\gamma - \#\beta = 0 \tag{2.63}$$

$$\phi[\mathcal{I}_+] : \sum_i \alpha_i^\phi = -\frac{k}{\alpha_+} \tag{2.64}$$

$$X[\mathcal{I}_+] : \sum_i \alpha_i^X = -i\sqrt{\frac{k}{2}} \tag{2.65}$$

$$Y[\mathcal{I}_+] : \sum_i \alpha_i^Y = 0 \tag{2.66}$$

whereas the laws implied by $\mathcal{I}_-$ are

$$\beta\gamma[\mathcal{I}_-] : \#\gamma - \#\beta = -k \tag{2.67}$$

$$\phi[\mathcal{I}_-] : \sum_i \alpha_i^\phi = -\frac{k}{\alpha_+} \tag{2.68}$$

$$X[\mathcal{I}_-] : \sum_i \alpha_i^X = +i\sqrt{\frac{k}{2}} \tag{2.69}$$

$$Y[\mathcal{I}_-] : \sum_i \alpha_i^Y = 0\,. \tag{2.70}$$

It is possible to find new conjugate vertex operators with respect to these conservation laws. In all these cases the operator conjugate to $V_{j,m,w}$ has in general a complicated form. The simplest expressions are found for the highest or lowest weight operators and they are

[8]Once more the conjugate representation is used. This is the natural thing to do for a $j=-1$ operator near the boundary.

[9]$m=0$ is required by regularity of the OPE with J^3, but the label in $V_{j,m,w}$ is actually $\widetilde{m} = m - \frac{k}{2}w$.

given by[10]

$$\widetilde{V}_{j,-j-1,w} = \beta^{k-2j-3} e^{\frac{2(2-k+j)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(-j-1)X} e^{i\sqrt{\frac{2}{k}}(-j-1+\frac{k}{2}w)Y} \tag{2.71}$$

$$\widetilde{V}^{+}_{j,j+1,w} = e^{\frac{2(j+1-\frac{k}{2})}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(j+1-\frac{k}{2})X} e^{i\sqrt{\frac{2}{k}}(j+1+\frac{k}{2}w)Y} \tag{2.72}$$

$$\widetilde{V}^{-}_{j,-j-1,w} = \gamma^{2j+2-k} e^{\frac{2(j+1-\frac{k}{2})}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(-j-1+\frac{k}{2})X} e^{i\sqrt{\frac{2}{k}}(-j-1+\frac{k}{2}w)Y} \tag{2.73}$$

More general vertex operators may be constructed from these ones applying the currents $J^{\pm}$. It is possible to obtain other vertices if one makes the change[11] $j \leftrightarrow -j-1$ in the above expressions. However, the choice that we made is dominant in the $\phi \to \infty$ limit.

How do these operators solve the winding non conservation problem? A generic correlation function in this new formalism is given by expectation values of the form (2.51), where now the vertex operator acting on the in vacuum is $V_{j,m,w}$ and the one acting on the out vacuum is a conjugate operator $\widetilde{V}_{j,m,w}$. This prescription specifies which realization of the identity is being used and, consequently, which conservation laws hold. However there is no natural choice for the intermediate vertex operators, and thus one can use either direct ($V_{j,m,w}$) or conjugate ($\widetilde{V}_{j,m,w}$) operators for the internal insertions. This last possibility allows to violate winding number conservation as follows.

Notice that the conservation laws for the fields X and Y associated with $\mathcal{I}_{\pm}$ amount to[12]

$$\begin{aligned} X[\mathcal{I}_{\pm}] : m_1 + \sum_{d.i.o.} m_i + \sum_{c.i.o.} \left(m_i \mp \frac{k}{2} \right) + m_n \mp \frac{k}{2} = \mp \frac{k}{2} \\ \to \sum_{i=1}^{n} m_i = \pm n_c \frac{k}{2} \end{aligned} \tag{2.74}$$

$$\begin{aligned} Y[\mathcal{I}_{\pm}] : \sum_{i=1}^{n} m_i + \frac{k}{2} \sum_{i=1}^{n} w_i = 0 \\ \to \sum_{i=1}^{n} w_i = \mp n_c \, , \end{aligned} \tag{2.75}$$

where the sum over *d.i.o.* is over *direct internal operators* and the sum over *c.i.o.* is over the *conjugate internal operators*. n_c is the number of *c.i.o.* in a correlation function, while $n - 2 - n_c$ is the number of *d.i.o.*

Equations (2.74) and (2.75) explicitly exhibit the amount of winding number non conservation of the correlators when internal conjugate operators are inserted. Furthermore the maximum total winding number of a correlator is $n-2$ since this is the maximum amount of internal operators. This result was suggested in [38] and demonstrated by algebraic arguments in [12] in the exact theory.

[10]Except for normalization factors that can be obtained from the two point functions

[11]Besides possible normalization factors

[12]Here only the conservation laws for X and Y are relevant since the others can be handled through the inclusion of screening operators.

To finish this section let us notice that there are other representations for the identity and conjugate operators. These are related via screening operators to other identities/operators in very much the same way that $\mathcal{I}_{-1}$ and 1 are. The existence of these representations is due to an accidental cancelation in the quadratic terms appearing in the conformal weight that allows to consider products of identities and screening operators. In this way one may construct yet another conjugate identity operator using the vertices (2.71). Indeed inserting the quantum numbers of the identity in the sector $w = 1$ one obtains

$$\widetilde{\mathcal{I}}_{+} = \beta^{-1} e^{-\frac{\alpha_+}{2}\phi} e^{-i\sqrt{\frac{k}{2}}X} \,. \tag{2.76}$$

This expression has to be defined through analytic continuation since negative powers of β cannot be understood otherwise. The same feature can be observed in the conjugate representations of vertices (2.72). Thus, we can consider alternate expressions

$$\widetilde{V}^{+}_{j,-j-1,w} = \beta^{-2j-2} e^{\frac{2(j+1-\frac{k}{2})}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(-j-1-\frac{k}{2})X} e^{i\sqrt{\frac{2}{k}}(-j-1+\frac{k}{2}w)Y} \,. \tag{2.77}$$

Once more the dominant expression in the $\phi \to \infty$ limit was chosen. This vertex was used in calculations in [21].

2.3.1 Screening operators in $w \neq 0$ sectors?

The reader might wonder at this point whether it is possible to incorporate winding number non-conservation in the Coulomb gas formalism in the usual way, i.e. through screening operators in $w \neq 0$ sectors. In that case one might avoid introducing conjugate vertices. This is the strategy we pursued in order to break the j conservation laws. However there are several arguments that imply this is not a possibility for winding number violation.

We observe that in order to violate winding conservation the hypothetical screening operators must be charged under X or Y which implies they should have an exponential factor in X or Y. It is interesting to note that the problem of finding all viable screening operators is dual to that of finding all possible interaction terms for the action (2.21) that do not break the original symmetries. For that reason we should require that these operators are not only $BRST$ invariant, but that they also have a full gauge invariant form that could be added to the action without gauge fixing.[13] This implies that we should not consider operators charged under Y for the inclusion of these in the action would break its symmetry under J^3. This leads us to consider operators charged under X. However, the requirement of gauge invariance poses another objection to the existence of these screening operators: it is not clear how to construct an operator with an exponential factor X that could have a gauge invariant form.[14]

Being difficult to work with full gauge invariant forms we forget about this problem and consider candidate $BRST$ invariant operators. Manifestly $BRST$ invariant operators

[13] See [33] for an explicit form of this action.

[14] Recall that the gauge field A is related to the field X through derivation.

that are charged under X have the form[15]

$$Q_1 = \beta^{b-m} e^{i\frac{2}{\alpha_+}b\phi} e^{i\sqrt{\frac{2}{k}}mX} \tag{2.78}$$

$$Q_2 = \gamma^{m-b} e^{i\frac{2}{\alpha_+}b\phi} e^{i\sqrt{\frac{2}{k}}mX}\ , \tag{2.79}$$

where $m + \frac{k}{2}w = 0$ and b is fixed by the condition of unit conformal weight. It turns out that their OPEs with $J^{\pm}$ are neither regular nor give total derivatives (as in the case of the screening operators (2.50)). This objection cannot be overcome by more sophisticated operators either, in particular by insertions of the form $\partial^n X$ or $(\partial X)^n$.

Finally, we could argue that if screening operators existed in $w \neq 0$ sectors it would be possible to violate winding number conservation by any amount. If one considers analytic continuation in the number of screening operators inserted in the correlators to any real or complex value (as required by the j conservation laws) it would be possible to violate winding conservation by an arbitrarily large non-integer number. However, as mentioned in the previous section, Maldacena and Ooguri proved in [12] that winding conservation of n-point functions can be violated by integer numbers bounded by $n-2$. This shows that consistency of the free field formalism requires the non-existence of screening operators in $w \neq 0$ sectors.

2.4 Two and three point amplitudes

Correlation functions in AdS_3 string theory have been computed in [12] using the exact results obtained in [18] for the $\frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)}$ coset. The Coulomb gas formalism was applied in [25] to calculate two and three point functions in the free field approximation to the $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$ model. The method was extended to $\mathrm{SL}(2,\mathbb{R})$ in [20, 21]. The results for winding conserving amplitudes in this approach agree with the exact ones. Three point functions violating winding conservation were originally computed in [21] and these results obtained in the free field approximation were later found in the exact theory in [12]. Here we briefly summarize a few aspects of the computations in [21] to facilitate the discussion of the supersymmetric case in the next section.

It turns out that it is easier to start with three point functions. Let us consider winding conserving amplitudes first. The simplest correlator contains one state of highest weight in the conjugate representation ($j \leftrightarrow -j-1$). Arbitrary three point amplitudes can be expressed as a function of this one acting with the lowering operator J^-. Indeed, applying J^- one gets correlators with the insertion of one state with $m = -1-j-N$, N being the number of lowering operators. After an analytic extension in N one gets any three point function.

Fixing as usual $z_1 = 0, z_2 = 1$ and $z_3 = \infty$, the calculation factorizes into correlators of $\beta\gamma$ fields and of exponential factors. Using screening operators $\mathcal{S}_+$, the first contribution amounts to

$$\left\langle \gamma^{j_2-m_2}(1)\gamma^{j_3-m_3}(\infty)\prod_{i=1}^{n_+}\beta(y_i)\right\rangle = \frac{\Gamma(-j_2+m_2+n_+)}{\Gamma(-j_2+m_2)}\prod_{i=1}^{n_+}|1-y_i|^{-1}\ , \tag{2.80}$$

where $n_+ = j_1 + j_2 + j_3 + 1$.

[15]We omit the surface integrals for the moment.

The exponential factors lead to integrals of the Dotsenko-Fateev type [35]. Putting all together, the three point amplitudes for states in arbitrary winding sectors are

$$\begin{aligned}\mathcal{A}_{\alpha_1,\alpha_2,\alpha_3} &= \delta^2(m_1+m_2+m_3)\frac{\Gamma(j_1-m_1+1)\Gamma(1+j_2-m_2)\Gamma(1+j_3-\bar{m}_3)}{\Gamma(-j_1+\bar{m}_1)\Gamma(-j_2+\bar{m}_2)\Gamma(m_3-j_3)}\times \\ &\times(k-2)\left[\pi\frac{\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right]^{n_+} D(j_1,j_2,j_3)\,, \end{aligned}\tag{2.81}$$

where $\alpha_i = j_i, m_i, \bar{m}_i, w_i$, $\sum_i w_i = 0$ and D is a function of j_i which is not necessary for our purposes here (see [21]).

The same procedure can be followed to compute correlators that do not preserve winding number conservation. To obtain amplitudes with $\sum_i w_i = +1(-1)$ one inserts a conjugate operator $\tilde{V}^+(\tilde{V}^-)$ at $z_2 = 1$ and performs the same steps. One gets (2.81) with $\delta^2(m_1+m_2+m_3) \to \delta^2(\pm\frac{k}{2}+m_1+m_2+m_3)$ and $D(j_1,j_2,j_3) \to D(j_1, 1+j_2-\frac{k}{2}, j_3)$.

The general form of the two point functions is dictated by conformal invariance. The two insertions must have the same conformal weight Δ (2.41) and verify the conservation laws (2.56) if direct (i.e. non conjugate) vertices are used. This leads to the following expression

$$\begin{aligned}&\langle V_{j_1,m,\bar{m},w}(z_1,\bar{z}_1)V_{j_2,-m,-\bar{m},-w}(z_2,\bar{z}_2)\rangle = \\ &\quad = |z_1-z_2|^{-4\Delta}\ \left[A(j_1,m,\bar{m})\ \delta(j_1+j_2+1) + B(j_1,m,\bar{m})\ \delta(j_1-j_2)\right]\,. \end{aligned}\tag{2.82}$$

Screening operators are not necessary to compute the first term and it is easy to see that $A(j_1,m,\bar{m}) = 1$.

The computation of $B(j_1,m,\bar{m})$ is more involved because screening operators have to be inserted. Moreover one cannot cancel the volume of the conformal group in (2.49) since only two points can be fixed.[16] Two techniques have been designed in [25] to deal with this term. One of them fixes the insertion points of the vertex operators at $z_1 = 0, z_2 = 1$ and of one of the screening operators at ∞; this cancels the full volume of the conformal group. The other one considers three point functions in the limit where the additional insertion goes to the identity ($j \to i0$). In the first case one gets

$$B(j,m,\bar{m}) = n_+\left(-\pi\frac{\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right)^{n_+}\frac{\Gamma(1-n_+)}{\Gamma(n_+)}\frac{\Gamma(-\frac{n_+}{k-2})}{\Gamma(1+\frac{n_+}{k-2})}\frac{\Gamma(j-m+1)\Gamma(1+j+\bar{m})}{\Gamma(-j-m)\Gamma(\bar{m}-j)}\,, \tag{2.83}$$

where we used the $\delta(j_1-j_2)$ to define $j = j_1 = j_2$ and $n_+ = 2j+1$. The result obtained by the second method differs from this one by $\frac{\delta(j_1-j_2)}{n_+}$.[17]

The same outcome is produced if one uses other conservation laws with the corresponding vertex operators.

[16] Actually this is also true for the first term $A(j_1,m,\bar{m})$. However we choose the normalization so that $A = 1$.

[17] This result contains an irrelevant factor $[\pi^2(k-2)]^{-n_+}$ with respect to reference [12]; notice that this factor is 1 when $n_+ = 0$, thus it does not affect the term $A(j,m,\bar{m})$.

3. Superstring theory on AdS_3

There is a direct extension of the WZW action to the supersymmetric case (see [39]) which can be written as

$$S_{\mathrm{SWZW}} = S_{\mathrm{WZW}} + S_f \tag{3.1}$$

where S_{WZW} is the bosonic WZW action and S_f is a free fermionic action. This is a surprising result leading to the conclusion that the supersymmetric WZW model can be decomposed into a bosonic part and a free fermionic theory. This interesting feature can be alternatively seen from a purely algebraic formulation of the theory. Indeed for SL$(2,\mathbb{R})$ one can generalize the OPEs (2.27)–(2.29) introducing a superfield $\mathcal{J}^a(z,\theta) = \psi^a(z) + \theta J^a(z)$ [40] where ψ denotes the supersymmetric partner of g, an element of a bosonic representation of the group. This verifies

$$\mathcal{J}^a(z_1,\theta_1)\mathcal{J}^b(z_2,\theta_2) \sim \frac{\frac{k}{2}\eta^{ab}}{(z_1-z_2)-(\theta_1-\theta_2)} + \frac{\theta_1-\theta_2}{z_1-z_2}\, i f^{ab}_c \mathcal{J}^c(z_2,\theta_2) \tag{3.2}$$

or equivalently, in components,

$$\psi^a(z)\psi^b(w) \sim \frac{\frac{k}{2}}{z-w}\eta^{ab} \tag{3.3}$$

$$J^a(z)\psi^b(w) \sim \psi^a(z)J^b(w) \sim \frac{i f^{ab}_c \psi^c(w)}{z-w} \tag{3.4}$$

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}}{(z-w)^2}g^{ab} + \frac{i f^{ab}_c J^c(w)}{z-w}\,. \tag{3.5}$$

The theory is thus equivalent to a bosonic *Kac-Moody* algebra for SL$(2,\mathbb{R})$ at level k and a fermionic *Kac-Moody* algebra of commuting currents at level k. For applications to string theory it is convenient to completely decouple both models. This is possible by defining

$$J^a(z) = j^a(z) - \frac{i}{k} f^a_{bc} : \psi^b(z)\psi^c(z) :\equiv j^a(z) + j^a_f(z)\,, \tag{3.6}$$

where $j(z)$ and $j_f(z)$ are bosonic currents leading to a *Kac-Moody* algebra at level $k+2$ and a free fermionic system respectively, for SL$(2,\mathbb{R})$.

It is now easy to construct the energy momentum tensor and the supersymmetry current as

$$T(z) = \frac{1}{k}\left(j^a(z)j_a(z) - \psi^a(z)\partial\psi_a(z)\right) \tag{3.7}$$

$$T_F(z) = \frac{2}{k}\left(\psi^a(z)j_a(z) - \frac{i}{3k} f_{abc}\psi^a(z)\psi^b(z)\psi^c(z)\right) \tag{3.8}$$

which form a superconformal $N=1$ theory with central charge

$$c_{\mathrm{SL}(2,\mathbb{R})} = \frac{3}{2} + \frac{3(k+2)}{k} \equiv \frac{3}{2} + \frac{3k'}{k'-2}\,, \tag{3.9}$$

where $k' \equiv k+2$.

In the previous section we mentioned the possibility of considering string propagation on $AdS_3 \times \mathcal{N}$. In critical bosonic string theory the internal manifold $\mathcal{N}$ allows to modify the dimension of spacetime, but it is not strictly necessary. However this issue is more subtle in the supersymmetric case where spacetime supersymmetry requires an internal theory. Actually it was shown in [34, 40] that spacetime supersymmetry requires $N = 2$ worldsheet supersymmetry. In particular it was observed in [40] that the coset $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$ possesses a natural complex structure allowing to enhance $N = 1$ to $N = 2$ supersymmetry. The problem is that it is not possible to directly extend this construction to $\mathrm{SL}(2,\mathbb{R})$ and this is the reason why one has to consider an internal manifold. Adding an internal theory makes it possible to dress the U(1) factor of $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$ with $N = 2$ supersymmetry.[18]

The general case was considered in references [45, 46] where the requirements to achieve spacetime supersymmetry were shown to be the following:

- $\mathcal{N}$ has to be a superconformal field theory (SCFT) with central charge
$$c_{\mathcal{N}} = 15 - c_{\mathrm{SL}(2,\mathbb{R})} = \frac{21}{2} - \frac{6}{k} \tag{3.10}$$
thus ensuring total central charge $c = 15$.
- $\mathcal{N}$ must possess an affine U(1) symmetry. Here χ will denote the supersymmetric partner of the $J^{\mathrm{U}(1)}$ current.
- The coset theory $\frac{\mathcal{N}}{\mathrm{U}(1)}$ must be $N = 2$ supersymmetric. The U(1) R-current of this model will be denoted $R^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$.

A consistent spacetime supersymmetric string theory sharing all these requisites can be built. It has at least $N = 2$ supersymmetry. In order to construct a theory with $N = 1$ spacetime supersymmetry one has to take a quotient by Z_2 [45]. Furthermore it was shown in [47] that these conditions are not only sufficient but they are also necessary to obtain supersymmetry.

Here we shall meet these minimal requirements using the least possible information about $\mathcal{N}$ so that our results will be very general. In the following section we shall develop the basic elements of this construction that will be necessary to obtain the vertex operators and compute correlation functions in this theory.

3.1 Spin fields, supercharges and vertex operators

In order to construct vertex operators it is convenient to bosonize the fermionic operators and currents. To work with canonically normalized scalars we redefine $\psi'^a \equiv \sqrt{\frac{2}{k}}\psi^a$ so that
$$\psi'^a(z)\psi'^b(w) \sim \frac{\eta^{ab}}{z-w}\,. \tag{3.11}$$

[18] Several examples have been considered in the literature. The study of *NS5-branes* leads to $AdS_3 \times S^3 \times T^4$ [10, 11, 41] which corresponds to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$ supersymmetric *WZW* model. The case $AdS_3 \times S^3 \times S^3 \times S^1$ [42], which is equivalent to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, has been reconsidered recently in the context of the AdS_3/CFT_2 duality in [43] (see also [44]). These theories present an extended $N = 4$ spacetime supersymmetry.

From now on we drop the primes to lighten the notation.

If only the three fermions in AdS_3 were available the bosonization could not be done. However the affine U(1) in $\mathcal{N}$ makes it possible to define canonically normalized bosons H_1, H_2 as [45, 48]

$$\partial H_1 \equiv \psi^1\psi^2 \qquad i\partial H_2 \equiv \psi^3\chi\,. \tag{3.12}$$

Similarly one can bosonize the currents $J^{\mathrm{U}(1)}$ and $R^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$ as [49]

$$J^{\mathrm{U}(1)} \equiv i\partial W \tag{3.13}$$

$$R^{\frac{\mathcal{N}}{\mathrm{U}(1)}} \equiv i\sqrt{\frac{c_{\mathcal{N}/\mathrm{U}(1)}}{3}}\partial Z = i\sqrt{3-\frac{2}{k}}\partial Z\,, \tag{3.14}$$

where the scalars W and Z are also canonically normalized.

The spin fields are constructed analogously to the flat case [49] as

$$S_r^+ = e^{ir(H_1-H_2)-\frac{i}{2}\sqrt{3-\frac{2}{k}}Z+i\sqrt{\frac{1}{2k}}W} \tag{3.15}$$

$$S_r^- = e^{ir(H_1+H_2)+\frac{i}{2}\sqrt{3-\frac{2}{k}}Z-i\sqrt{\frac{1}{2k}}W}\,, \tag{3.16}$$

where $r = \pm\frac{1}{2}$. Notice that there are two distinct spin fields S^+ and S^- as required by $N = 2$ spacetime supersymmetry. In fact supercharges are constructed as usual [48], namely

$$Q_r^\pm = (2k)^{\frac{1}{4}}\oint \mathrm{d}z\, e^{-\frac{\varphi}{2}}S_r^\pm\,, \tag{3.17}$$

where φ denotes the bosonization of the ghost fields.

The presence of W and Z in the spin fields is unusual. W provides the supercharges (3.17) with the correct charge under the U(1) current of spacetime *R-symmetry* [49]

$$\mathcal{R} = \sqrt{2k}\oint \mathrm{d}z\, J^{\mathrm{U}(1)}\,, \tag{3.18}$$

whereas the field Z carries information about the supersymmetry of $\frac{\mathcal{N}}{\mathrm{U}(1)}$. In fact it was shown in [47] that Z can be identified with the scalar bosonizing the current $R^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$.

We now have all the ingredients to construct the superstring vertex operators. Since this theory is equivalent to the product of $\mathrm{SL}(2,\mathbb{R})_{k+2}$ times a free fermionic theory, the vertex operators can be expected to factorize accordingly. In particular the vertex operators of the bosonic theory will represent states of the superstring as well, taking into account that the quantum numbers are determined by the purely bosonic currents j^a. This implies that one has to replace $k \to k' = k+2$ in the expressions of section 2.2. In this section we will rely, mostly, on what has been done in [49].

Let us start by considering the states with zero winding number in the *Neveu-Schwarz* sector (NS). The most general expression for the ground state operators is

$$\mathcal{V}^{-1}_{j,m,q,h} = e^{-\varphi}e^{iqW}V_{\frac{\mathcal{N}}{\mathrm{U}(1)}}V_{j,m}\,, \tag{3.19}$$

where the superindex denotes the ghost charge, h is the conformal weight of the operator in $\frac{\mathcal{N}}{\mathrm{U}(1)}$ and q is the charge under $\mathcal{R}$.

These operators have to satisfy two physical state conditions. First they must be on mass-shell, i.e.

$$\Delta\left(\mathcal{V}^{-1}_{j,m,q,h}\right)=\frac{1}{2}+\frac{q^2}{2}+h-\frac{j(j+1)}{k}=1\,. \tag{3.20}$$

Second, they must survive the *GSO* projection or, equivalently, they must be mutually local with the supercharges (3.17). This constraint implies

$$q\sqrt{\frac{2}{k}}-q_R\in 2\mathbb{Z}+1\,, \tag{3.21}$$

where q_R is the worldsheet charge of $V_{\frac{\mathcal{N}}{\mathrm{U}(1)}}$ under $R^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$. This expression exhibits the relevance of the internal theory to achieve spacetime supersymmetry.

Let us now discuss the construction of vertex operators at excitation level $\frac{1}{2}$. Since the vertices must comprise a realization of the full algebra (with currents J^a given by (3.6)) the fields ψ^a must combine with $V_{j,m}$ so that they belong to a representation of J^a. This was done in [41] where the following combinations were found[19]

$$\begin{aligned}
(\psi V_j)_{j+1,m} &\equiv 2(j+1-m)(j+1+m)\psi^3 V_{j,m}+(j+m)(j+1+m)\psi^+ V_{j,m-1}+\\
&\quad+(j-m)(j+1-m)\psi^- V_{j,m+1} && (3.22)\\
(\psi V_j)_{j,m} &\equiv 2m\psi^3 V_{j,m}-(j+m)\psi^+ V_{j,m-1}+(j-m)\psi^- V_{j,m+1} && (3.23)\\
(\psi V_j)_{j-1,m} &\equiv 2\psi^3 V_{j,m}-\psi^+ V_{j,m-1}-\psi^- V_{j,m+1}\,. && (3.24)
\end{aligned}$$

Here the external (internal) subindex is the eigenvalue under J^a (j^a). It was also shown in [41] that the combination (3.23) is not *BRST* invariant and thus there are two ways to combine the fermionic excitations in vertex operators. Considering for example (3.24), one obtains

$$\mathcal{W}^{-1}_{j,m,q,h}=e^{-\varphi}e^{iqW}V_{\frac{\mathcal{N}}{\mathrm{U}(1)}}\left(\psi V_j\right)_{j-1,m}\,. \tag{3.25}$$

The mass shell condition is now

$$\Delta\left(\mathcal{W}^{-1}_{j,m,q,h}\right)=\frac{1}{2}+\frac{q^2}{2}+h-\frac{j(j+1)}{k}+\frac{1}{2}=1+\frac{q^2}{2}+h-\frac{j(j+1)}{k}=1\,, \tag{3.26}$$

and the *GSO* projection implies

$$q\sqrt{\frac{2}{k}}-q_R\in 2\mathbb{Z}\,. \tag{3.27}$$

The same procedure can be applied to construct the operator $\mathcal{X}^{-1}_{j,m,q,h}$ from the combination (3.22).

The construction of vertex operators in the *Ramond* sector is analogous. One now has to build representations of J^a from combinations of $S_r^{\pm}$ and $V_{j,m}$. In this analysis the contributions to $S_r^{\pm}$ from the fields W and Z are irrelevant since they can be absorbed into

[19] There is a numerical factor, regarding normalization, that differs from the one used in [41]. This difference affects correlation functions. However, numerical factors non-depending on j and m are not important for this work.

redefinitions of e^{iqW} and $V_{\frac{\mathcal{N}}{\mathrm{U}(1)}}$. Thus one may write

$$\left(S^{\pm}V_j\right)_{j-\frac{1}{2},m} \equiv S^{\pm}_{\frac{1}{2}}V_{j,m-\frac{1}{2}} - S^{\pm}_{-\frac{1}{2}}V_{j,m+\frac{1}{2}}\,. \tag{3.28}$$

The combinations possessing S^+ and S^- are related through conjugation. They represent states with different charges under $\mathcal{R}$ (3.18). There is of course another representation of spin fields which couples to $j+\frac{1}{2}$ denoted $(S^{\pm}V_j)_{j+\frac{1}{2},m}$.

Following the same procedure as in the NS sector one may obtain the vertex operator in the *picture* $-\frac{1}{2}$, namely

$$\mathcal{Y}^{-\frac{1}{2}(\pm)}_{j,m,q,h} = e^{-\frac{\varphi}{2}}e^{iqW}V_{\frac{\mathcal{N}}{\mathrm{U}(1)}}\left(S^{\pm}V_j\right)_{j-\frac{1}{2},m}\,. \tag{3.29}$$

The mass shell condition is

$$\Delta\left(\mathcal{Y}^{-\frac{1}{2}(\pm)}_{j,m,q,h}\right) = 1 \mp \frac{q_R}{2} + \frac{q^2}{2} \pm \frac{q}{\sqrt{2k}} + h - \frac{j(j+1)}{k} = 1\,, \tag{3.30}$$

and the GSO projection implies

$$q\sqrt{\frac{2}{k}} - q_R \in 2\mathbb{Z}+1\,. \tag{3.31}$$

One can proceed similarly with $(S^{\pm}V_j)_{j+\frac{1}{2},m}$.

This completes the study of supersymmetric vertex operators in the zero winding sector for the lowest excitation levels. It is now necessary to introduce states in nonzero winding sectors. Similarly as in the bosonic theory one can act on the states of the $w=0$ sector with the spectral flow operator. In order to construct this operator in the supersymmetric theory one might take the exponential of the scalar bosonizing the current $J^3 = j^3 - i\partial H_1$. However the operator defined as in (2.45) is not mutually local with the supercharges (3.17). So, one includes a *twist* in the U(1) factor of the model (actually, this is a natural consequence of the complex structure of the factor $\mathrm{U}(1)^2$, that is required by supersymmetry[20]). Therefore the generalized spectral flow operator is given by

$$\mathcal{F}^w_{\pm} = e^{iw\sqrt{\frac{k}{2}}(\Upsilon \pm W)}\,, \tag{3.32}$$

where the field Υ bosonizes J^3. Notice that there are two possible spectral flow operators, but it will turn out that only one of them is necessary to generate all the states in the theory.

The operator (3.32) may be rewritten in terms of the fields bosonizing j^3 (Y) and the fermions (H_1) as follows

$$\mathcal{F}^w_{\pm} = e^{iw\left(\sqrt{\frac{k+2}{2}}Y + H_1 \pm \sqrt{\frac{k}{2}}W\right)}. \tag{3.33}$$

[20] This is the factor one obtains when formulating the theory on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)^2 \times \frac{\mathcal{N}}{\mathrm{U}(1)}$.

Note that the spectral flow operation can relate states of type $\mathcal{V}$ with others of type $\mathcal{W}$ or $\mathcal{X}$ depending on whether one acts with $\mathcal{F}^w_+$ or $\mathcal{F}^w_-$ [49]. Thus only one of them will generate new states. In the *Ramond* sector these operators generate states related by conjugation.

Following the procedure discussed in section 2.2.1 one can construct vertex operators in $w \neq 0$ sectors. These are given by

$$\mathcal{V}^{-1,w}_{j,m,q,h} = e^{-\varphi} e^{i\left(q+w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} V^{\mathrm{susy}}_{j,m,w} \tag{3.34}$$

$$\mathcal{W}^{-1,w}_{j,m,q,h} = e^{-\varphi} e^{i\left(q-w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} \left(\psi V^{\mathrm{susy}}_{j,w}\right)_{j-1,m} \tag{3.35}$$

$$\mathcal{Y}^{-\frac{1}{2},w(\pm)}_{j,m,q,h} = e^{-\frac{\varphi}{2}} e^{i\left(q-w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} \left(S^{\pm} V^{\mathrm{susy}}_{j,w}\right)_{j-\frac{1}{2},m}, \tag{3.36}$$

where we defined

$$\begin{aligned} V^{\mathrm{susy}}_{j,m,w} &= V_{j,m} e^{i\sqrt{\frac{2}{k'}}mX} e^{i\sqrt{\frac{2}{k'}}(m+\frac{k'}{2}w)Y} e^{iwH_1} \\ &= \gamma^{j-m} e^{\frac{2j}{\alpha'_+}\phi} e^{i\sqrt{\frac{2}{k'}}mX} e^{i\sqrt{\frac{2}{k'}}(m+\frac{k'}{2}w)Y} e^{iwH_1} \end{aligned} \tag{3.37}$$

($\alpha'_+ = \sqrt{2k'-4} = \sqrt{2k}$), which has conformal dimension

$$\Delta\left(V^{\mathrm{susy}}_{j,m,w}\right) = -\frac{j(j+1)}{k} - \frac{k}{4}w^2 - mw\,. \tag{3.38}$$

The mass shell and *GSO* projection conditions can be obtained as usual.

This concludes the study of vertex operators in the supersymmetric theory. We now turn to a discussion of the scattering amplitudes.

3.2 Supersymmetric Coulomb gas formalism

The extension of the formalism described in section 2.3 to the supersymmetric case is straightforward. Since the supersymmetric model is the product of a bosonic theory and a free fermionic theory one would expect that screening operators, identities and conjugate vertices could be built similarly as in the bosonic model. We will do this in a constructive manner in order to show that there are no other possibilities given by supersymmetry that may render the formalism inconsistent.

Let us start by discussing the screening operators. Recall that the symmetry currents $\mathcal{J}^a$ are given by (3.2), so that requiring commutation with them is equivalent to demanding regular OPEs with both J^a and ψ^a. This implies that the screening operators must not include fermionic fields and consequently they will contribute to the OPE with J^a only through contractions with j^a (3.6). This is an important result. Since the currents j^a realize a level k' *Kac-Moody* algebra, the screening operators in the supersymmetric theory coincide with those in the bosonic theory replacing $k \to k'$, namely

$$\mathcal{S}'_+ = \int d^2z\, \beta(z)\overline{\beta}(\overline{z}) e^{-\frac{2}{\alpha'_+}\phi(z,\overline{z})} \tag{3.39}$$

$$\mathcal{S}'_- = \int d^2z\, \beta(z)^{\frac{\alpha'^2_+}{2}} \overline{\beta}(\overline{z})^{\frac{\alpha'^2_+}{2}} e^{-\alpha'_+\phi(z,\overline{z})}\,. \tag{3.40}$$

It is easy to verify that these operators have zero conformal dimension.

Therefore the conservation laws can be modified in the same way it was discussed for the bosonic case. Let us stress that these screening operators do not allow to alter the conservation laws either of the fermionic fields or of the fields in the internal theory.

What about the identity operators?

One can proceed as in the bosonic case, applying the spectral flow operator on the identity and then replacing $j \to \frac{k}{2} - 2 - j$. Since the identity is not in the physical spectrum it will not necessarily be written as one of the vertex operators described in the previous section. Actually the operator 1 is still of the form $V_{j,m}$ with $m = 0$ and $j = -1$ as in the bosonic case. This raises several observations. First, as discussed for the screening operators, the condition of regularity of the OPE with the currents implies that the identity operator must not contain fermionic fields. Second, this condition must hold for the symmetry currents of the full spacetime, that is $AdS_3 \times \mathrm{U}(1) \times \frac{\mathcal{N}}{\mathrm{U}(1)}$. Thus the identity cannot be charged under the field W or depend on χ. Finally, the natural form of the identity 1 is in the picture 0. These comments indicate that a good starting point to find new representations of the identity operator is the spectral flow of $V_{j,m}$. This is as in the bosonic case except that the spectral flow operator (3.32) now has contributions from the U(1) factor as well as from the fermions. Therefore the general form of the candidate conjugate identity operator is

$$\mathcal{I}_* = e^{iaH_1} e^{i(q \pm \sqrt{\frac{k}{2}} w)W} V^{\mathrm{susy}}_{j,m,w} \tag{3.41}$$

with $w = \pm 1$. Notice that the zero modes of H_1 and W are shifted in the same way that the field Y was shifted for the bosonic case. This is interpreted as the string winding around $AdS_3 \times \mathrm{U}(1)$. Consequently the quantum numbers a, m and q are actually tilded variables (similarly as m in the bosonic case). But once again we omit the tildes in the operators.

Regular OPEs with the fermions $\psi^\pm$ imply[21] $a + w = 0$, while the OPEs with the U(1) symmetry currents χ and ∂W determine that there cannot be fermionic contributions from U(1) and $q \pm \sqrt{\frac{k}{2}} w = 0$ respectively. These constraints lead to the following expression

$$\mathcal{I}_* = e^{-iwH_1} V^{\mathrm{susy}}_{j,m,w} = \gamma^{j-m} e^{\frac{2j}{\alpha'_+}\phi} e^{i\sqrt{\frac{2}{k'}} mX} e^{i\sqrt{\frac{2}{k'}}(m + \frac{k'}{2} w)Y}. \tag{3.42}$$

Finally, plugging in the quantum numbers of the identity $m + \frac{k'}{2} w = 0$ and $j = \frac{k'}{2} - 1$, one obtains the same operators as in the bosonic theory with the replacement $k \to k'$, namely

$$\mathcal{I}'_+ = e^{-\frac{k'}{\alpha'_+}\phi} e^{-i\sqrt{\frac{k'}{2}} X} \tag{3.43}$$

$$\mathcal{I}'_- = \gamma^{-k'} e^{-\frac{k'}{\alpha'_+}\phi} e^{i\sqrt{\frac{k'}{2}} X}. \tag{3.44}$$

The conservation laws dictated by these operators coincide with the bosonic ones except for the change $k \to k'$. The same conclusion holds for the conjugate identities $\mathcal{I}_{-1}$, $\widetilde{\mathcal{I}}_0$ and $\tilde{\mathcal{I}}_+$.

[21] Here we use the explicit expression (3.37) for $V^{\mathrm{susy}}_{j,m,w}$.

Analogously to the bosonic case one can write conjugate vertex operators with respect to these identities. One has only to replace the factor $V_{j,m,w}$ by its conjugate versions (2.71)–(2.73), thus obtaining

$$\widetilde{\mathcal{V}}^{*-1,w}_{j,m,q,h} = e^{-\varphi} e^{i\left(q+w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} e^{iwH_1} \widetilde{V'_{j,m,w}}^{\,*} \tag{3.45}$$

$$\widetilde{\mathcal{W}}^{*-1,w}_{j,m,q,h} = e^{-\varphi} e^{i\left(q-w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} \left(\psi e^{iwH_1} \widetilde{V'_{j,w}}^{\,*}\right)_{j-1,m} \tag{3.46}$$

$$\widetilde{\mathcal{Y}}^{*-\frac{1}{2},w(\pm)}_{j,m,q,h} = e^{-\frac{\varphi}{2}} e^{i\left(q-w\sqrt{\frac{k}{2}}\right)W} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} \left(S^{\pm} e^{iwH_1} \widetilde{V'_{j,w}}^{\,*}\right)_{j-\frac{1}{2},m}, \tag{3.47}$$

where $\widetilde{V'_{j,m,w}}^{\,*}$ refers to any of (2.71)–(2.73)[22] with $k \to k'$.

To finish this section let us notice that conjugation involves only the bosonic part of AdS_3. Therefore the associated conservation laws cannot break the conservation of the quantum number q or the fermionic excitations. This fact has important consequences that are not observed in the bosonic case. In particular, correlation functions for states with different excitation numbers will exhibit a different winding number violation pattern. While some choice of winding violation is nonvanishing for one case, it will vanish identically for other cases, involving excited states. We shall ellaborate on this point in the following sections.

3.3 Supersymmetric correlators: the Neveu-Schwarz sector

In this section we compute correlation functions in the Neveu-Schwarz sector, both satisfying and violating winding number conservation.

3.3.1 Two point functions in the Neveu-Schwarz sector

The two point functions can be computed by either one of the two methods discussed in section 2.3. Since the insertion that one should consider in order to obtain the result as the limit of a three point function is purely bosonic, both methods turn out to be identical.

Recall that the ghost charge of the correlators must be -2. Therefore we can use the natural form of the vertex operators in the picture -1. Moreover the direct version of the vertices can be taken since violation of winding number conservation is not expected. There are thus three different types of two point functions one can consider; those with: two insertions of $\mathcal{V}$, two insertions of $\mathcal{W}$ and one insertion of each kind.

We shall omit the contributions from the internal $\frac{\mathcal{N}}{\mathrm{U}(1)}$ theory and thus the computations correspond to an $AdS_3 \times \mathrm{U}(1)$ background.

Let us start by considering the correlation functions of two ground states $\mathcal{V}$:

$$\left\langle \mathcal{V}^{-1,w_1}_{j_1,m_1,q_1}(z_1)\, \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2}(z_2) \right\rangle = \left\langle e^{-\varphi(z_1)} e^{-\varphi(z_2)} \right\rangle \left\langle e^{i\left(q_1+w_1\sqrt{\frac{k}{2}}\right)W(z_1)} e^{i\left(q_2+w_2\sqrt{\frac{k}{2}}\right)W(z_2)} \right\rangle \times$$
$$\times \left\langle e^{iw_1H_1(z_1)} e^{iw_2H_2(z_2)} \right\rangle \left\langle V'_{j_1,m_1,w_1}(z_1) V'_{j_2,m_2,w_2}(z_2) \right\rangle . \tag{3.48}$$

[22] At least for highest weight operators; the others can be obtained applying lowering modes of the currents.

Note that this factorization is possible since the screening operators in the supersymmetric theory coincide with those in the bosonic model. Fixing as usual $z_1 = 0$ and $z_2 = 1$, we obtain

$$\left\langle \mathcal{V}^{-1,w_1}_{j_1,m_1,q_1}(0)\,\mathcal{V}^{-1,w_2}_{j_2,m_2,q_2}(1)\right\rangle = \langle V'_{j_1,m_1,w_1}(0)V'_{j_2,m_2,w_2}(1)\rangle\,, \tag{3.49}$$

where the conservation laws for W and H_1 are the following

$$W: \quad q_1 + q_2 + \sqrt{\frac{k}{2}}\,(w_1 + w_2) = 0 \tag{3.50}$$

$$H_1: \quad w_1 + w_2 = 0 \tag{3.51}$$

The two point functions (3.49) were computed in the free field approximation in [21] and the result (2.82) is quoted in section 2. Here we only have to change $k \to k'$.

Let us now consider two point functions of type $\langle \mathcal{W}\mathcal{W}\rangle$, namely

$$\left\langle \mathcal{W}^{-1,w_1}_{j_1,m_1,q_1}(z_1)\mathcal{W}^{-1,w_2}_{j_2,m_2,q_2}(z_2)\right\rangle = \left\langle e^{-\varphi(z_1)}e^{-\varphi(z_2)}\right\rangle \left\langle e^{i\left(q_1 - w_1\sqrt{\frac{k}{2}}\right)W(z_1)}e^{i\left(q_2 - w_2\sqrt{\frac{k}{2}}\right)W(z_2)}\right\rangle \times$$
$$\times \left\langle \left(\psi V^{\text{susy}}_{j_1,w}\right)_{j_1-1,m_1}(z_1)\left(\psi V^{\text{susy}}_{j_2,w}\right)_{j_2-1,m_2}(z_2)\right\rangle \tag{3.52}$$

Here the contribution from the field W leads to

$$W: \quad q_1 + q_2 - \sqrt{\frac{k}{2}}\,(w_1 + w_2) = 0\,. \tag{3.53}$$

In order to deal with the terms (ψV) it is convenient to bosonize the fermions as $\psi^+ = \sqrt{2}e^{iH_1}; \psi^- = \sqrt{2}e^{-iH_1}$ which leads to

$$T^+ \equiv \psi^+ e^{iwH_1}V'_{j,m-1,w} = \sqrt{2}e^{i(w+1)H_1}V'_{j,m-1,w} \tag{3.54}$$

$$T^- \equiv \psi^- e^{iwH_1}V'_{j,m+1,w} = \sqrt{2}e^{i(w-1)H_1}V'_{j,m+1,w} \tag{3.55}$$

Notice that this implies the conservation laws $w_1 + w_2 \pm 2 = 0$ from the field H_1 for the terms $\langle T^\pm T^\pm\rangle$, whereas the factors $V'_{j,m,w}$ give $w_1 + w_2 = 0$. Therefore the only non-vanishing contributions arise from contractions of ψ^3 and from terms containing $\langle T^\pm T^\mp\rangle$. The commutation relations $[J_0^\pm, V_{j,m}] = (m \mp j)V_{j,m\pm1}$ can be used to replace the vertex operators $V_{j,m+1}$ or $V_{j,m-1}$ and after a little algebra one obtains

$$\left\langle \mathcal{W}^{-1,w_1}_{j_1,m_1,q_1}\,\mathcal{W}^{-1,w_2}_{j_2,m_2,q_2}\right\rangle = \frac{4j(2j+1)}{m^2 - j^2}\,\langle V'_{j_1,m_1,w_1}V'_{j_2,m_2,w_2}\rangle \tag{3.56}$$

for $j_1 = j_2 = j$ and $m_1 = -m_2 \equiv m$.[23]

Lastly let us discuss the mixed correlator $\langle \mathcal{V}\mathcal{W}\rangle$. It is easy to see that this vanishes identically. Indeed the term with ψ^3 leaves an unpaired fermion whereas the conservation laws from the field H_1 and the operators $V_{j,m,w}$ are incompatible.

This concludes the analysis of the two point functions in the NS sector. In the next section we compute three point functions.

[23]The same result was found in [50].

3.3.2 Three point functions in the Neveu-Schwarz sector

Here there are additional complications to take into account. Firstly the ghost charge asymmetry implies that one has to consider one operator in the picture 0. Secondly the three point functions can in principle violate winding number conservation and thus one has to consider different conservation laws.

In order to obtain the vertex operator in the picture 0 we have to construct the picture changing operator. Generalizing the standard procedure followed in the flat theory, where the picture changing operator is $P_{+1} = e^{\varphi} T_F$, we find for string theory in AdS_3

$$P_{+1}^{AdS_3} = \frac{e^{\varphi}}{\sqrt{2k}} \left\{ -2\psi^3 j^3 + \psi^+ j^- + \psi^- j^+ + \psi^- \psi^+ \psi^3 \right\} , \tag{3.57}$$

where the supercorrent T_F was defined in (3.8).

To obtain the picture changing operator in the full theory one has to add $T_F^{\mathrm{U}(1)}$ and $T_F^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$. The first term is easy to write since the U(1) WZW model is a flat theory, thus

$$T_F^{\mathrm{U}(1)} = \chi \partial W . \tag{3.58}$$

In order to define $T_F^{\frac{\mathcal{N}}{\mathrm{U}(1)}}$ we have to choose a particular internal theory. However this is not necessary for the computation of the three point functions of ground states in the NS sector since they do not contain fermionic contributions (either from χ or from any fermion in $\frac{\mathcal{N}}{\mathrm{U}(1)}$) and thus their contraction with T_F gives one fermion in the picture 0 vertex operator that cannot be paired. Therefore we can define

$$\mathcal{V}^0(z) = \lim_{w \to z} P_{+1}^{AdS_3}(w) \mathcal{V}^{-1}(z) \tag{3.59}$$

and thus

$$\begin{aligned} \mathcal{V}_{j,m,q}^{0,w} = \frac{e^{i\left(q + \sqrt{\frac{k}{2}} w\right) W}}{\alpha'_+} \Big\{ & -2m\psi^3 e^{iwH_1} V'_{j,m,w} + \sqrt{2}(m-j) e^{i(w-1)H_1} V'_{j,m+1,w} + \\ & + \sqrt{2}(m+j) e^{i(w+1)H_1} V'_{j,m-1,w} \Big\} . \end{aligned} \tag{3.60}$$

Similarly for $\mathcal{W}$ we obtain[24]

$$\begin{aligned} \mathcal{W}_{j,m,q}^{0,w=0} = \sqrt{\frac{2}{k}} e^{iqW} \{ & k' \partial\gamma V'_{j,m+1,w=0} - j\psi^- \psi^3 V'_{j,m+1,w=0} \\ & - j\psi^3 \psi^+ V'_{j,m-1,w=0} - j\psi^+ \psi^- V'_{j,m,w=0} \} . \end{aligned} \tag{3.61}$$

We can now compute the three point functions. Let us start with correlators of the type $\langle \mathcal{V}\mathcal{V}\mathcal{V} \rangle$. Since the vertex operators $\mathcal{V}^{-1}$ do not contain ψ^3 the first term in (3.60) will

[24] In this case we consider states in the $w = 0$ sector. General states can be obtained in a similar way considering contractions from the fermionic part of the spectral flow operator. We leave the details to the reader.

not contribute. There are thus only two terms to consider, namely[25]

$$\left\langle \mathcal{V}^{-1,w_1}_{j_1,m_1,q_1}\,\mathcal{V}^{0,w_2}_{j_2,m_2,q_2}\,\mathcal{V}^{-1,w_3}_{j_3,m_3,q_3}\right\rangle =$$

$$= \frac{\sqrt{2}}{\alpha'_+}\left\langle e^{i\left(q_1+\sqrt{\frac{k}{2}}w_1\right)W} e^{i\left(q_2+\sqrt{\frac{k}{2}}w_2\right)W} e^{i\left(q_3+\sqrt{\frac{k}{2}}w_3\right)W}\right\rangle \times$$

$$\times\left\{(m_2-j_2)\left\langle e^{iw_1H_1}e^{i(w_2-1)H_1}e^{iw_3H_1}\right\rangle\langle V'_{j_1,m_1,w_1}V'_{j_2,m_2+1,w_2}V'_{j_3,m_3,w_3}\rangle+\right.$$

$$\left.+(m_2+j_2)\left\langle e^{iw_1H_1}e^{i(w_2+1)H_1}e^{iw_3H_1}\right\rangle\left\langle V'_{j_1,m_1,w_1}V'_{j_2,m_2-1,w_2}V'_{j_3,m_3,w_3}\right\rangle\right\}, \quad (3.62)$$

where we fix as usual $z_1=\infty$, $z_2=0$ and $z_3=1$. The conservation law associated to the field W is the following for both terms

$$q_1+q_2+q_3+\sqrt{\frac{k}{2}}\,(w_1+w_2+w_3)=0 \quad (3.63)$$

whereas the field H_1 gives

$$w_1+w_2+w_3\mp 1=0 \quad (3.64)$$

with the $-$ $(+)$ sign corresponding to the first (second) term in (3.62). This is a very interesting result. In fact notice that the correlator $\langle\mathcal{VVV}\rangle$ is nonvanishing only if the winding number is not conserved. Moreover the expressions (3.63) and (3.64) show that non trivial contributions from the factor U(1) other than those arising from the spectral flow sector are necessary to obtain nonvanishing correlators.

Therefore the explicit computation of (3.62) requires a conjugate operator in the internal position either according to $\mathcal{I}_+$ or to $\mathcal{I}_-$ and only one term will contribute in each of these cases. For example using $\mathcal{I}_-$ one obtains

$$\left\langle \mathcal{V}^{-1,w_1}_{j_1,m_1,q_1}\,\mathcal{V}^{0,w_2}_{j_2,m_2,q_2}\,\mathcal{V}^{-1,w_3}_{j_3,m_3,q_3}\right\rangle = \frac{\sqrt{2}}{\alpha'_+}(m_2-j_2)\left\langle V'_{j_1,m_1,w_1}\widetilde{V'}^{-}_{j_2,m_2+1,w_2}V'_{j_3,m_3,w_3}\right\rangle \quad (3.65)$$

with the following conservation laws

$$m_1+m_2+m_3+1+\frac{k}{2}=0 \quad (3.66)$$

$$w_1+w_2+w_3-1=0\,. \quad (3.67)$$

Notice that these conditions are compatible with (3.64). The same procedure can be followed for $\mathcal{I}_+$. In this case winding conservation will be violated by one unit with the opposite sign and the other term will survive in (3.62).

Using the explicit form of the bosonic correlator (2.81) it is easy to rewrite (3.65) as

$$\left\langle \mathcal{V}^{-1,w_1}_{j_1,m_1,q_1}\,\mathcal{V}^{0,w_2}_{j_2,m_2,q_2}\,\mathcal{V}^{-1,w_3}_{j_3,m_3,q_3}\right\rangle = \frac{\sqrt{2}}{\alpha'_+}\left\langle V'_{j_1,m_1,w_1}\widetilde{V'}^{-}_{j_2,m_2,w_2}V'_{j_3,m_3,w_3}\right\rangle \quad (3.68)$$

and then the amplitudes of type $\langle\mathcal{VVV}\rangle$ coincide with the bosonic ones.

[25] In the following expressions a $\frac{\sqrt{2}}{\alpha'_+}$ factor appears. This is related to the supercurrent and it can be absorbed after properly normalizing the picture 0 vertex operators. We disregard this normalization for it is irrelevant for our purposes.

Here it is interesting to comment on an observation by Giveon and Kutasov [7]. They suggested that correlators of operators $\mathcal{V}$ should present a natural framework to violate winding conservation in the coset model $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$. This feature is related in their work to the picture changing operator, i.e. the $-\frac{1}{2}$ mode of the supercurrent T_F. Actually, because of the $N=2$ supersymmetry, one can decompose $G_{-\frac{1}{2}}$ into eigenstates under the *R-symmetry* current. Then one can choose the term with either positive or negative charge. This gives two possible picture changing operators each one affecting the m conservation law in $\pm\frac{k}{2}$ and from this one can read the violation to winding conservation. We have presented here the first explicit calculation of this fact in the free field formalism. It is also interesting to notice that in the formalism we have developed one can read the two options proposed by Giveon and Kutasov for the picture changing operator in (3.62) and observe that only one of them contributes to a given correlator. In order for any of these two correlation functions to be non-zero it is necessary to guarantee that the bosonic part of the correlator will have a set of conservation laws that match those that are obtained in the fermionic part. This is possible, in the free field formalism, only because of the existence of the conjugate identities in the $w=\pm1$ winding sectors. This is true not only for the coset but for the full $\mathrm{SL}(2,\mathbb{R})$ WZW model.

This concludes the computation of the three point functions of type $\langle\mathcal{V}\mathcal{V}\mathcal{V}\rangle$. The same procedure can be extended to the other three point functions. For example $\langle\mathcal{V}\mathcal{W}\mathcal{V}\rangle$ and $\langle\mathcal{V}\mathcal{W}\mathcal{W}\rangle$. The structure of the winding violation pattern is more complicated as one adds excited fields of type $\mathcal{W}$. However, it is easy to see that the result is zero in the last case unless w conservation is violated, analogously to the case of $\langle\mathcal{V}\mathcal{V}\mathcal{V}\rangle$. On the other hand the function $\langle\mathcal{V}\mathcal{W}\mathcal{V}\rangle$ presents all the possibilities regarding w conservation.

Considering amplitudes of type $\langle\mathcal{W}\mathcal{W}\mathcal{W}\rangle$ introduces a difficulty related to the factor $\partial\gamma$ in $\mathcal{W}^0$. However the formalism developed in [25] to compute $\beta\gamma$ correlators can be easily adapted to deal with this case. It is interesting, though, that in this particular case the supersymmetric correlation functions are not proportional to the bosonic ones involving only ground states, but have factors proportional to derivatives. This is analogous to what happens in flat space.

3.4 Supersymmetric Correlators: the Ramond sector

In this section we discuss correlation functions involving states in the Ramond sector. These have not been considered previously in the literature. We compute amplitudes in arbitrary winding sectors and consider the possibility of violating this quantum number.

3.4.1 Two point functions in the Ramond sector

In order to achieve ghost charge -2 in the two point functions in the Ramond sector we need vertex operators in the *picture* $-\frac{3}{2}$. The non trivial BRST invariant vertices are[26]

$$\mathcal{Y}^{-\frac{3}{2}(\pm)}_{j,m,q,h} = e^{-\frac{3\varphi}{2}} e^{iqW} V_{\frac{\mathcal{N}}{\mathrm{U}(1)}} \left(S^{-\frac{3}{2},\pm} V_j\right)_{j-\frac{1}{2},m}\ , \tag{3.69}$$

[26]These can be obtained applying considerations discussed in [45] and [51].

where

$$S_r^{-\frac{3}{2},\pm} = e^{ir(H_1 \pm H_2) \mp \frac{i}{2}\sqrt{3-\frac{2}{k}}Z \pm i\sqrt{\frac{1}{2k}}W} . \tag{3.70}$$

Therefore we have to compute the correlator

$$\begin{aligned}
&\left\langle \mathcal{Y}_{j_1,m_1,q_1}^{-\frac{3}{2},w_1,(\pm)}(z_1)\mathcal{Y}_{j_2,m_2,q_2}^{-\frac{1}{2},w_2,(\pm)}(z_2)\right\rangle = \\
&\quad = \left\langle e^{-\frac{3\varphi(z_1)}{2}} e^{-\frac{\varphi(z_2)}{2}}\right\rangle \times \\
&\quad \times \left\langle e^{i\left(q_1-\sqrt{\frac{k}{2}}w_1\right)W}\left(S^{-\frac{3}{2},\pm}V_{j_1}^{w_1}\right)_{j_1-\frac{1}{2},m_1}(z_1) e^{i\left(q_2-\sqrt{\frac{k}{2}}w_2\right)W}\left(S^{\pm}V_{j_2}^{w_2}\right)_{j_2-\frac{1}{2},m_2}(z_2)\right\rangle .
\end{aligned}$$

The conservation law for the field H_2 implies that terms containing spin fields of the same (opposite) type ($\pm$), namely $S_r^{-\frac{3}{2},\pm}$ and $S_{r'}^{\pm}$, vanish if $r \neq r'$ ($r = r'$). Similarly the conservation law from H_1 implies that correlators contaning spin fields of the same type must violate winding number in one unit if $r = r'$. Therefore two point functions involving operators of the same type vanish since winding number must be conserved. We stress that the H_2 field is responsible for inhibiting this channel. We thus compute the nontrivial correlator, where winding conservation equals H_2 conservation,

$$\begin{aligned}
&\left\langle \mathcal{Y}_{j_1,m_1,q_1}^{-\frac{3}{2},w_1,(\pm)}\,\mathcal{Y}_{j_2,m_2,q_2}^{-\frac{1}{2},w_2,(\mp)}\right\rangle = \\
&= -\left\langle e^{i\left(q_1-\sqrt{\frac{k}{2}}w_1 \pm \frac{1}{\alpha'_+}\right)W} e^{i\left(q_2-\sqrt{\frac{k}{2}}w_3 \mp \frac{1}{\alpha'_+}\right)W}\right\rangle \times \\
&\quad \times \left\{\left\langle e^{\pm\frac{i}{2}H_2} e^{\mp\frac{i}{2}H_2}\right\rangle \left\langle e^{i\left(+\frac{1}{2}+w_1\right)H_1} e^{i\left(-\frac{1}{2}+w_2\right)H_1}\right\rangle \left\langle V'_{j_1,m_1-\frac{1}{2},w_1} V'_{j_2,m_2+\frac{1}{2},w_2}\right\rangle + \right. \\
&\quad\quad \left. + \left\langle e^{\mp\frac{i}{2}H_2} e^{\pm\frac{i}{2}H_2}\right\rangle \left\langle e^{i\left(-\frac{1}{2}+w_1\right)H_1} e^{i\left(+\frac{1}{2}+w_2\right)H_1}\right\rangle \left\langle V'_{j_1,m_1+\frac{1}{2},w_1} V'_{j_2,m_2-\frac{1}{2},w_2}\right\rangle \right\}
\end{aligned}$$

with the following conservation laws

$$q_1 + q_2 - \sqrt{\frac{k}{2}}\,(w_1 + w_2) = 0 \tag{3.71}$$

$$w_1 + w_2 = 0 \,. \tag{3.72}$$

We can again use the commutator $[J_0^+ V_{j,m}]$ in the last factor obtaining

$$\left\langle \mathcal{Y}_{j_1,m_1,q_1}^{-\frac{3}{2},w_1,(\pm)}\,\mathcal{Y}_{j_2,m_2,q_2}^{-\frac{1}{2},w_2,(\mp)}\right\rangle = \frac{2m}{j+\frac{1}{2}-m}\left\langle V'_{j_1,m_1-\frac{1}{2},w_1} V'_{j_2,m_2+\frac{1}{2},w_2}\right\rangle , \tag{3.73}$$

where $j = j_1 = j_2$ and $m = m_1 = -m_2$.

There is an alternative way to perform this computation. Mimicking the bosonic calculation one can take the limit of a three point function containing an identity operator. However inserting an operator of the type $\mathcal{V}_{j,m}$ and taking the limit $j \to -1$, $m \to 0$ produces a factor $e^{-\varphi}$ which is clearly not the identity (actually this operator has conformal dimension $\frac{1}{2}$). On the other hand, an operator of the type $V_{j,m}$ gives the identity when taking the appropriate limit but it does not satisfy the ghost charge condition. We propose to proceed as follows. There is a representation of the identity in the supersymmetric theory which is very useful, namely

$$\mathcal{I}^{-2}(z) = e^{-2\varphi(z)}\,. \tag{3.74}$$

So we can apply the picture changing operator to one of the R vertices and compute

$$\left\langle \mathcal{Y}^{w_1,(\pm)}_{j_1,m_1,q_1}\, \mathcal{Y}^{w_2,(\pm)}_{j_2,m_2,q_2} \right\rangle = \left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1}\, e^{-2\varphi}\, \mathcal{Y}^{\frac{1}{2},w_2,(\pm)}_{j_2,m_2,q_2} \right\rangle. \tag{3.75}$$

We now move on to three point functions.

3.4.2 Three point functions in the Ramond sector

The simplest three point function one may consider is of the form $\langle \mathcal{YVY} \rangle$. Notice that in this case the charge asymmetry condition can be satisfied when all the vertex operators take their natural form. Therefore the correlation function is as follows

$$\begin{aligned}
&\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1}\, \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2}\, \mathcal{Y}^{-\frac{1}{2},w_3,(\pm)}_{j_3,m_3,q_3} \right\rangle = \\
&= \left\langle e^{-\frac{\varphi(z_1)}{2}} e^{-\varphi(z_2)} e^{-\frac{\varphi(z_3)}{2}} \right\rangle \left\langle e^{i\left(q_1-\sqrt{\frac{k}{2}}w_1\right)W} \left(S^{\pm} V^{w_1}_{j_1}\right)_{j_1-\frac{1}{2},m_1}(z_1) e^{i\left(q_2+\sqrt{\frac{k}{2}}w_2\right)W} \times \right. \\
&\qquad \left. \times V^{w_2}_{j_2,m_2}(z_2) e^{i\left(q_3-\sqrt{\frac{k}{2}}w_3\right)W} \left(S^{\pm} V^{w_3}_{j_3}\right)_{j_3-\frac{1}{2},m_3}(z_3) \right\rangle.
\end{aligned}$$

The conservation law for the field H_2 gives nonvanishing results from terms containing either $S^{\pm}_{-\frac{1}{2}}$ and $S^{\pm}_{\frac{1}{2}}$ or $S^{\pm}_{\pm\frac{1}{2}}$ and $S^{\mp}_{\pm\frac{1}{2}}$. Therefore if both vertices $\mathcal{Y}$ are of the same type ($+$ or $-$) one gets

$$\begin{aligned}
&\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1}\, \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2}\, \mathcal{Y}^{-\frac{1}{2},w_3,(\pm)}_{j_3,m_3,q_3} \right\rangle = \\
&= \left\langle e^{i\left(q_1-\sqrt{\frac{k}{2}}w_1\pm\frac{1}{\alpha'_+}\right)W} e^{i\left(q_2+\sqrt{\frac{k}{2}}w_2\right)W} e^{i\left(q_3-\sqrt{\frac{k}{2}}w_3\pm\frac{1}{\alpha'_+}\right)W} \right\rangle \times \\
&\quad \times \left\{ \left\langle e^{\mp\frac{i}{2}H_2} e^{\pm\frac{i}{2}H_2} \right\rangle \left\langle e^{i\left(+\frac{1}{2}+w_1\right)H_1} e^{iw_2 H_1} e^{i\left(-\frac{1}{2}+w_3\right)H_1} \right\rangle \times \right. \\
&\quad \times \left\langle V'_{j_1,m_1-\frac{1}{2},w_1} V'_{j_2,m_2,w_2} V'_{j_3,m_3+\frac{1}{2},w_3} \right\rangle + \left\langle e^{\pm\frac{i}{2}H_2} e^{\mp\frac{i}{2}H_2} \right\rangle \times \\
&\quad \left. \times \left\langle e^{i\left(-\frac{1}{2}+w_1\right)H_1} e^{iw_2 H_1} e^{i\left(+\frac{1}{2}+w_3\right)H_1} \right\rangle \left\langle V'_{j_1,m_1+\frac{1}{2},w_1} V'_{j_2,m_2,w_2} V'_{j_3,m_3-\frac{1}{2},w_3} \right\rangle \right\}
\end{aligned}$$

with the following conservation laws

$$q_1+q_2+q_3+\sqrt{\frac{k}{2}}\left(-w_1+w_2-w_3\right) \pm \sqrt{\frac{2}{k}} = 0 \tag{3.76}$$

$$w_1+w_2+w_3 = 0\,. \tag{3.77}$$

It is interesting that these correlators cannot violate winding number conservation. Moreover this condition arises from the conservation law of the field H_2 which bosonizes the fermions ψ^3 and χ. Thus the U(1) factor determines which correlators are nonvanishing.

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Finally, if conditions (3.76) and (3.77) are verified, the correlator becomes

$$\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1} \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2} \mathcal{Y}^{-\frac{1}{2},w_3,(\pm)}_{j_3,m_3,q_3} \right\rangle = \Big\{ \left\langle V'_{j_1,m_1-\frac{1}{2},w_1} V'_{j_2,m_2,w_2} V'_{j_3,m_3+\frac{1}{2},w_3} \right\rangle + \\ + \left\langle V'_{j_1,m_1+\frac{1}{2},w_1} V'_{j_2,m_2,w_2} V'_{j_3,m_3-\frac{1}{2},w_3} \right\rangle \Big\} . \quad (3.78)$$

If the internal vertex $\mathcal{V}$ is annihilated by J_0^+, one can insert such operator and rewrite the correlators as

$$\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1} \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2} \mathcal{Y}^{-\frac{1}{2},w_3,(\pm)}_{j_3,m_3,q_3} \right\rangle_{S^2} = \frac{j_3 - j_1 + m_1 - m_3}{m_1 - \frac{1}{2} - j_1} \times \\ \times \left\langle V'_{j_1,m_1-\frac{1}{2},w_1} V'_{j_2,m_2,w_2} V'_{j_3,m_3+\frac{1}{2},w_3} \right\rangle . \quad (3.79)$$

There is an equivalent expression if the internal operator corresponds to a lowest weight state.

The case where the operators $\mathcal{Y}$ are not of the same type (i.e. one is $+$ and the other one is $-$) can be treated similarly. The novelty is that in this case the correlation function must violate w conservation. The conservation laws imply that each one of the terms in the correlator must satisfy different conditions, namely

$$\left(S^+_{\frac{1}{2}} S^-_{\frac{1}{2}} \right) : w_1 + w_2 + w_3 + 1 = 0 \quad (3.80)$$

$$\left(S^+_{-\frac{1}{2}} S^-_{-\frac{1}{2}} \right) : w_1 + w_2 + w_3 - 1 = 0 \quad (3.81)$$

and besides, in both cases,

$$q_1 + q_2 + q_3 + \sqrt{\frac{k}{2}} (-w_1 + w_2 - w_3) = 0 . \quad (3.82)$$

Proceeding as before one obtains for the first condition (3.80)

$$\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1} \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2} \mathcal{Y}^{-\frac{1}{2},w_3,(\mp)}_{j_3,m_3,q_3} \right\rangle = \left\langle V'_{j_1,m_1-\frac{1}{2},w_1} \widetilde{V'}^{+}_{j_2,m_2,w_2} V'_{j_3,m_3-\frac{1}{2},w_3} \right\rangle \quad (3.83)$$

whereas for the second one (3.81) we find

$$\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1} \mathcal{V}^{-1,w_2}_{j_2,m_2,q_2} \mathcal{Y}^{-\frac{1}{2},w_3,(\mp)}_{j_3,m_3,q_3} \right\rangle = \left\langle V'_{j_1,m_1+\frac{1}{2},w_1} \widetilde{V'}^{-}_{j_2,m_2,w_2} V'_{j_3,m_3+\frac{1}{2},w_3} \right\rangle . \quad (3.84)$$

The last correlator we shall consider is $\langle \mathcal{Y}\mathcal{W}\mathcal{Y} \rangle$. This case is interesting since there is a direct contribution from the field H_2. The pattern of winding (non) conservation is contrary to the previous one, i.e. w conservation is violated in the case where the vertices are both of type $+$ or of type $-$ whereas it is conserved if they are of opposite type.

Unlike in the NS sector here the term containing ψ^3 also contributes. It is convenient to rewrite the bosonization (3.12) as

$$\chi \pm \psi^3 = \sqrt{2} e^{\pm i H_2} . \quad (3.85)$$

Thus we consider

$$\left\langle \mathcal{Y}^{-\frac{1}{2},w_1,(\pm)}_{j_1,m_1,q_1}\, \mathcal{O}^{-1,w_2}_{j_2,m_2,q_2}\, \mathcal{Y}^{-\frac{1}{2},w_3,(\pm)}_{j_3,m_3,q_3} \right\rangle \tag{3.86}$$

where

$$\mathcal{O}^{-1,w_2}_{j_2,m_2,q_2} = \sqrt{2} e^{-\varphi} e^{i\left(q_2 - \sqrt{\frac{k}{2}} w_2\right)} \left(e^{iH_2} - e^{-iH_2}\right) e^{iw_2 H_1} V'_{j_2,m_2,w_2} \,. \tag{3.87}$$

The term e^{iH_2} (e^{-iH_2}) selects factors containing $S^{\pm}_{\pm\frac{1}{2}}$ ($S^{\pm}_{\mp\frac{1}{2}}$). Some of them conserve winding and others have total winding ± 1, therefore all the possibilities are present in this case. This is due to the field H_2. The explicit computation is similar to the cases discussed previously. The reader can easily fill up the details.

4. Conclusions

The original motivation of this work was to extend the Coulomb gas formalism for string theory on AdS_3 to the supersymmetric case. We would like to stress that the formalism presented in section 3.2 was developed constructively. This indicates that not only is it possible to extend the bosonic formulation of [20, 21] but also that the supersymmetric Coulomb gas formalism designed from scratch gives the proper extension. This is very important to assure the uniqueness of the basic objects in the business: screening operators and identities. Actually we have considered and discarded the possibility of constructing new operators of this sort in the supersymmetric theory.

In particular, we argued against the existence of screening operators in $w \neq 0$ sectors and consequently the formalism naturally obeys the winding non-conservation pattern of the bosonic theory shown by Maldacena and Ooguri in [12]. Moreover this general pattern is preserved in the supersymmetric theory. However, it was found that, due to selection rules related to the fermionic part of the theory, some channels are inhibited in this case. Thus, the possibility of violating winding number conservation is dependent on the excitation number of the operators involved in the supersymmetric correlation functions.

The method was employed to compute two and three point functions of physical states in both Neveu-Schwarz and Ramond sectors. Correlators of Neveu-Schwarz states in different winding sectors, both obeying and violating winding number conservation were presented. We found, as expected, that the supersymmetric correlators can be expressed in terms of the corresponding bosonic ones.

Furthermore we explicitly computed two and three point correlators in the Ramond sector. We analyzed the structure of the pattern of violation to winding conservation and stressed the important role played in this matter by the conservation laws of the field H_2, related to the U(1) factor of the theory.

Important problems remain. Above all the computation of four point functions. Even though the method we have presented is only an approximation, valid near the boundary of spacetime, we expect that if this is a consistent model this approach will exhibit the factorization properties of a unitary theory.

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